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MATHEMATICAL PRINCIPLES OF FRACTANCE APPROXIMATION CIRCUITS AND THEIR APPLICATIONS

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Abstract: This article is written to commemorate the 100th anniversary of the birth of Rashid Shakirovich Nigmatullin. He first realized the fractal element performing the differentiation and integration operation of the half-order in electrochemistry in the early 1960s. In recent years, as the theory and application of fractional calculus has become a hot topic in many fields, the circuit modeling and mathematical modeling of complex fractional order phenomena and processes, as well as the physical realization and practical applications of fractional order circuits and systems are particularly important and urgent. Designing and constructing fractance approximation circuits (FACs) are an effective technique to realize fractional operators and fractional elements. In this article, we will introduce and discuss the research and development of FACs. The main contents are: 1) some pioneers in the research of FACs; 2) basic concepts of fractional-order circuit elements and FACs;3) Oldham fractal chain circuits and their mathematical descriptions, some classical half-order fractal FACS; 4) mathematical basis of the frequency-domain analysis—frequency-domain characteristics and operational characteristics; 5) Liu-Kaplan fractal chain circuits and their mathematical descriptions; 6) scaling extension theory and irregular scaling equations.

Keywords: fractional element, fractional operator, fractance, fractor, fractal fractance approximation circuits, operational characteristics, analogical circuit modeling, mathematical rational approximation, scaling extension, irregular scaling equations.

1. "Fractional Calculus" Theory and Application： Pioneers-Heroes Timing List①

This is a historical course map (Fig.1.1) that I started making around 2011. At that time, I was writing a monograph, "Mathematical Principles of Fractance Approximation Circuits" [1], which involved the collection of historical documents related to the theory and applications of fractional calculus, and the need to organize and briefly summarize the research and development history of these fields.

In this course of historical development, the brave pioneers are the most important. The efforts and contributions of the pioneers are worth remembering and honoring. Therefore, I later named this course map as "'Fractional Calculus' Theory and Applications: Pioneers-Heroes Timing List", and it was published in the "Annotated translation preface" of my annotated translation book "Fractional Calculus: Theoretical Fundamentals and Introduction to Applications" [2]. This book is translated from Igor Podlubny's "Fractional Differential Equations" [3].

In this historical course map, Rashid Shakirovich Nigmatullin has a clear place. His outstanding contributions and achievements in many aspects, especially the discovery and realization of half-order fractal elements in electrochemistry [4-11], have established his solid position in the development and application of fractional calculus.

This is also an evolutionary map, that I have had to show many times every year in my classroom teaching to the graduate students of related majors, in the College of Electronics and

① In this section personal pronoun (I) coincides with the name of the leading author Yuan Xiao.

Information Engineering of Sichuan University (in Chengdu, China) and the College of Information Science and Technology of Tibet University (in Lhasa, China) for more than ten years.

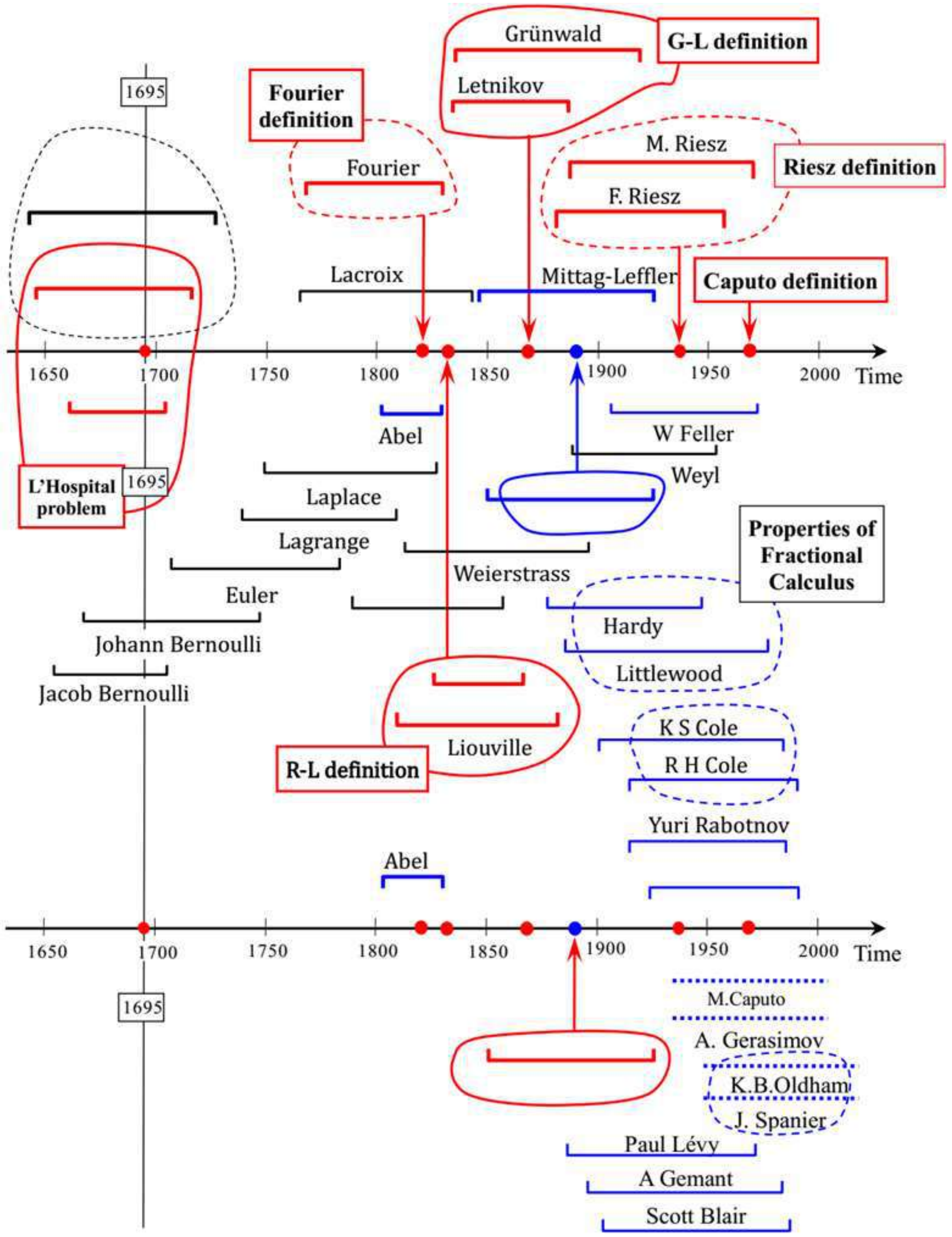


Fig. 1. “Fractional Calculus” Theory and Applications: Pioneers-Heroes Timing List

I often tell my students that in their study and scientific research, they should learn from these brave pioneers, and have the courage to discover, to open up, to explore, and to innovate. It is possible for researchers in any discipline to embark on the path of fractional-order and achieve results. In particular, I will certainly mention the discoveries and contributions in the fractional-order field, which were made by not-math-major-born researchers, such as Keith B. Oldham, Rashid Shakirovich Nigmatullin, and others. Tell the stories of these pioneers, inspire students' enthusiasm, enlighten students' minds. So I also tell my students: All roads lead to fractional-order!

2. Fractional-order circuit elements and fractance approximation circuits: basic concepts

2.1. Fractional-order circuit elements: Circuit symbols and their mathematical representations

▪ Fractional-order element

Fractional-order circuit element (→FOE), or simply fractional element, is a class of circuit elements or devices with fractional order differintegration capability (also known as operational capability). That is, these devices are two-terminal devices with fractional-order impedance (or admittance), and multi-terminal devices with fractional-order transfer functions (i.e., fractional-order system functions).

Fractional-order impedance or admittance → Fractional-order immittance → Fractance [12].

The simplest fractional element is the passive two-terminal fractional element, which we call fractor. Fractor is a passive two-terminal fractional element with fractional-order immittance.

▪ Ideal fractor and non-ideal fractor

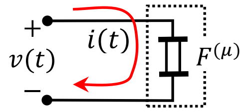
The input impedance function of an ideal μ-order fractor is a fractance function, which is defined by

$$Z^{(\mu)}(s) = \frac{V(s)}{I(s)} = I^{(\mu)}(s) = F^{(\mu)}s^{\mu}, \tag{1}$$

where μ is the operational order ($0 < |\mu| < 1$, $\mu \in \mathbb{R}$), $s = \sigma + \mathrm{j}\Omega$ is the operational variable (also called complex frequency variable, or Laplacian variable), $F^{(\mu)}$ is the lumped parameter of the element which is called fractance quantity, referred to as fractance.

For an ideal fractor, $F^{(\mu)}$ is a constant independent of the operational variable. We use the circuit symbol shown in fig. 2*(a)* to represent the ideal fractor [1, 2]. The circuit symbol and its definition for the non-ideal fractor is shown in fig. 2*(b)*.

Ideal μ-order fractor



Circuit symbol

Input impedance—Ideal μ-order fractance function

$$Z^{(\mu)}(s) = \frac{V(s)}{I(s)} = I^{(\mu)}(s) = F^{(\mu)}s^{\mu}$$

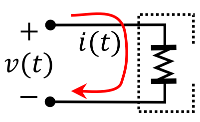
SI unit: $[F^{(\mu)}] = \Omega\mathrm{s}^{\mu}$

SI dimension: $\mathrm{ML^2T^{-3+\mu}I^{-2}}$

Lumped parameter $F^{(\mu)}, 0 < |\mu| < 1$ ←Fractance

(a) Ideal fractor: circuit symbol and its definition, SI dimension and SI unit

Non-ideal μ-order fractor



Circuit symbol

Input impedance

$$\tilde{Z}^{(\mu)}(s) = \frac{V(s)}{I(s)} = \tilde{I}^{(\mu)}(s) \approx I^{(\mu)}(s)$$

Non-ideal μ-order fractance function $\tilde{I}^{(\mu)}(s)$ is an irrational function that contains or approximately contains fractional operator with order μ (i.e., s^{μ}), which is not strictly equal to the ideal fractance function $I^{(\mu)}(s) = F^{(\mu)}s^{\mu}$.

(b) Non-ideal fractor: circuit symbol and its definition

Fig. 2. Fractors: circuit symbols, the input impedance functions

▪ Inductive fractor and capacitive fractor

A fractor which the operational order μ in the open interval $(-1, 0)$ is called a capacitive fractor, and is often called a fractional capacitor. And in contrast to this, is the inductive fractor, $\mu \in (0, 1)$.

Up to now, both ideal fractors and wideband non-ideal fractors are almost unavailable devices.

Therefore, a very natural idea is to use easily available integer-order elements, such as resistors, capacitors, inductors, and active devices, and so on, to build a finite size two-terminal circuit network in a certain frequency band to approximate the operational function of an ideal fractor. We call this a fractance approximation circuit, abbreviated as FAC [1, 2].

2.2. Some pioneers in the research of Fractance Approximation Circuits

In Fractance Approximation Circuit (→FAC) field, the following researchers have made pioneering contributions, which deserve our memory and highly esteem. They are

★1920s Oliver Heaviside: Discovered the $-1/2$ order RC distribution cable;

★1959 R.Morrison: Scaling fractal ladder, fractal chain, and two fractal chain circuits [13];

★1960s R Sh Nigmatullin: “recond” and “reind”, physically fabricated FOE [4-11];

★1960s G.E.Carlson: Carlson fractal lattice circuit, Carlson regular iterating approximation [14,15];

★1960s S.C.Dutta Roy: Distributed & lumped realizations, Circuit modeling by CFE and PFE [16,17];

★1970s K.B.Oldham: Semiintegral electroanalysis, half-order fractal chain circuit [18];

★1985 S.H.Liu, T.Kaplan: Liu fractal tree [19], Liu-Kaplan fractal chain circuit [20-21];

★1992 M.Nakagawa, K.Sorimachi: N-S fractal tree circuit [22];

★1997 C.Haba et al: Haba’s MOS fractal capacitors [23-25]

……

2.3. Fractance Approximation Circuits (FACs): Mathematical Descriptions

Figure 3 shows the relationship between the FACs and the fractors, and its approximation process. This one-port passive (or active) circuit network with finite size $k \in \mathbb{N}$ is called a Fractance Approximation Circuit (FAC), which approximates to the ideal fractor in a certain frequency range and under a given approximation precision.

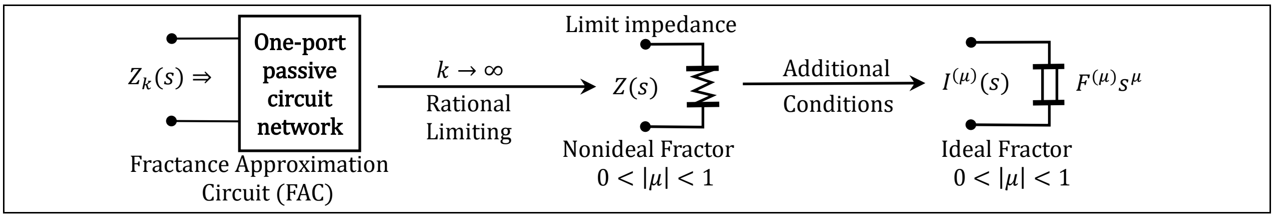


Fig. 3. Fractance approximation Circuit and fractors—Mathematical principle of the FACs

In mathematical terms, a sequence of rational impedance functions $\{Z_k(s)\}_{k\in\mathbb{N}}$ is constructed to converge to the limit impedance function $Z(s)$. This limit impedance function is equaled to the irrational function $I^{(\mu)}(s)$ of an ideal fractor directly or under some additional conditions (such as in the low or high frequency range), namely,

$$Z_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{n_k} b_{k,i} s^i}{\sum_{i=0}^{d_k} a_{k,i}\, s^i} \xrightarrow[\text{Rational limiting}]{k\to\infty} Z(s) \xrightarrow[\text{Conditions}]{\text{Additional}} I^{(\mu)}(s) = F^{(\mu)} s^{\mu}, \qquad (2)$$

where k is the number of FAC size or of iterations of an approximation algorithm, is also the complexity of a FAC whose value is natural number: $k \in \mathbb{N}$, n_k and d_k are the highest degrees of the numerator and denominator polynomials, $N_k(s)$ and $D_k(s)$ respectively.

2.4. Basic mathematical properties of the impedance function sequences

The input impedance function sequence of a FAC and the rational approximation sequence of a fractional operator, $\{Z_k(s)\}_{k\in\mathbb{N}}$, must satisfy the following basic mathematical properties [1, 2].

1) **Computational Rationality**

For the operational variable (that is, the independent variable) s in the impedance function $Z_k(s)$, there can only be addition, subtraction, multiplication, division of four rational operations, there should be no irrational operations. This is because it is necessary to avoid constructing new fractance approximation circuits by use of fractors.

2) **Positive Reality Principle**

This is a necessary condition for a causally stable system. That is, the basic requirements of physical realization. Specifically, all its zeros and poles should be located in the left half plane of the complex plane s.

3) **Operational Validity—Convergence and Limit Impedance Function**

The rational function sequence converges and has

$$\lim_{k\to\infty} Z_k(s) = Z(s) \approx I^{(\mu)}(s) = F^{(\mu)} s^{\mu} \quad 0 < |\mu| < 1\,. \tag{3}$$

The limit impedance function $Z(s)$ must have fractional-order operational performance at least in a certain frequency range (or band). Therefore, the limit impedance or admittance function must be an irrational function that contains or approximately contains fractional operator with order μ (i.e., s^{μ}).

The operational validity is the core problem of the fractance approximation circuit!

Mathematically, the operational validity is supported by convergence and limit immittance.

- **Ideal approximation and non-ideal approximation**

An approximation where the limit impedance function $Z(s)$ is equaled to the ideal fractance function $I^{(\mu)}(s)$, is called an ideal approximation. That is $Z(s) = I^{(\mu)}(s)$.

An approximation that requires additional conditions (such as in high or low frequency range) to make the above equation true or approximately equal, are called a non-ideal approximations. That is $Z(s) \approx I^{(\mu)}(s)$.

- **Strong approximation and weak approximation**

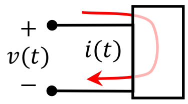
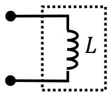
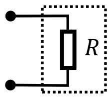
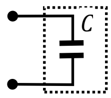
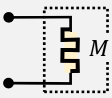
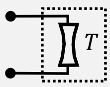
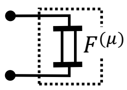
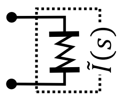
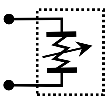
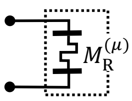
$$Z_k(s) = \frac{N_k(s)}{D_k(s)} = \kappa \frac{\sum_{i=0}^{n_k-1}(s - z_i)}{\sum_{i=0}^{d_k-1}(s - p_j)}, \qquad k \in \mathbb{N}^+ \tag{4}$$

If all zeros z_i and poles p_j of the rational approximation function $Z_k(s)$, lie on the negative real axis of the operational complex plane s, $z_i \in \mathbb{R}^-, i = 0\sim n_k - 1$, $p_j \in \mathbb{R}^-, j = 0\sim d_k - 1$, then this approximation is called a strong approximation, otherwise $\mathrm{Re}z_i \in \mathbb{R}^-, i = 0\sim n_k - 1$, $\mathrm{Re}p_j \in \mathbb{R}^-, j = 0\sim d_k - 1$, the rational approximation is called a weak approximation.

2.5. Passive integer-order elements and fractional-order elements: circuit symbols and Mathematical descriptions

Circuit symbols and their mathematical descriptions of some of the passive integer-order and fractional-order circuit elements are listed in Table 1.

Table 1. Integer-order elements and fractional-order elements: Circuit symbols and mathematical descriptions

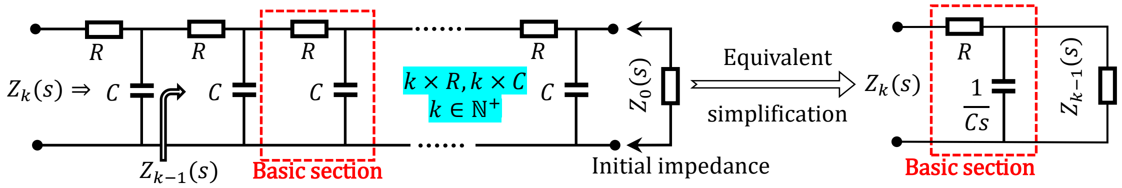
Element name	Circuit symbol	Lumped parameter	Impedance function $Z(s)$	Operational order μ	Element Classification	Remark
Ideal two terminals element			$Z(s)=\dfrac{V(s)}{I(s)}$	$\Lambda(\varpi)=\lg\lvert Z(\mathrm{j}\cdot 10^{\varpi})\rvert$ $\Theta(\varpi)=\arg\{Z(\mathrm{j}\cdot 10^{\varpi})\}$ $M(\varpi)=\dfrac{\mathrm{d}\Lambda(\varpi)}{\mathrm{d}\varpi}$		$\mathrm{j}=\sqrt{-1}$
Ideal Inductor		Inductance L $\mathrm{d}\varphi=L\mathrm{d}i$	$Z_L(s)=Ls$ $F^{(+1)}=L$	$\mu=+1$		$\Lambda(\varpi)$: Amplitude-frequency characteristics, or magnitude-frequency characteristics
Ideal Resistor		Resistance R $\mathrm{d}v=R\mathrm{d}i$	$Z_L(s)=R$ $=Rs^0$ $F^{(0)}=R$	$\mu=0$		$\Theta(\varpi)$: Phase-frequency characteristics $M(\varpi)$: Order-frequency characteristics
Ideal Capacitor		Capacitance C $\mathrm{d}q=C\mathrm{d}v$	$Z_C(s)=\dfrac{1}{Cs}$ $F^{(-1)}=C^{-1}$	$\mu=-1$	Integer-order element	
Memristor [26-28]		Memristance M $\varphi=M\mathrm{d}q$	??	??		Four fundamental circuit variables:
Ideal Transtor [29, 30]		Transtance T $\varphi=T\mathrm{d}q$	??	??		Voltage v Current i Charge q Magnetic-flux φ (flux-linkage)
Ideal μ-order fractor[1]		Ideal Fractance $F^{(\mu)}$	$Z^{(\mu)}(s)=F^{(\mu)}s^{\mu}$ $I^{(\mu)}(s)=F^{(\mu)}s^{\mu}$	$0<\lvert\mu\rvert<1$		SI unit: $\left[F^{(\mu)}\right]=\Omega s^{\mu}$ SI dimension: $\mathrm{Dim}F^{(\mu)}$ $=\mathrm{ML^2T^{-3+\mu}I^{-2}}$
Nonideal μ-order fractor		Nonideal Fractance $\tilde{F}$?	$\tilde{I}^{(\mu)}(s)\approx I^{(\mu)}(s)$ Additional condition		Fractional-order element	$0<\mu<1$: Inductive fractors $-1<\mu<0$: Capacitive fractors
Variable-order fractor[31]		??	??	??		
Frac-memristor [32-34]		??	??	??		
Notes	Basic circuit variables: $v,\ i,\ q,\ \varphi.$		Operational variable: $s=\sigma+\mathrm{j}\Omega$ Frequency exponent: $\varpi=\lg\Omega \Leftrightarrow \Omega=10^{\varpi}$			

3. Oldham fractal chain circuits and their mathematical description

Oldham K. B. and Spanier J., in their famous book “Fractional Calculus” [17], proposed and studied a Fractance Approximation Circuit with negative half-order operational performance in a low-frequency range. In order to express our deep respect for Oldham’s pioneering research work in the development and application of fractional calculus, we call such circuit Oldham type I fractal chain circuit, or simply Oldham fractal chain circuit.

3.1. Prototype circuit and iterating circuit

The original circuit first proposed by Oldham et al is shown in fig. 4*(a)*. Obviously, we can equivalently simplify this circuit to the iterating form shown in fig. 4*(b)*, and call it an iterating circuit.



(a) Prototype circuit *(b)* Iterating circuit

Fig. 4. Oldham type I fractal chain circuit

3.2. Iterating algorithm and iterating function—Mathematization of the problem

Given any physically realizable rational initial impedance $Z_0(s) = N_0(s)/D_0(s)$, the input impedance function sequence $\{Z_k(s)\}_{k\in\mathbb{N}}$ of Oldham type I fractal chain circuit, can be found from the iterating algorithm formula:

$$Z_0(s) = \frac{N_0(s)}{D_0(s)} \to Z_k(s) = F_0\big(Z_{k-1}(s)\big), \qquad k \in \mathbb{N}^+. \tag{5}$$

Where $F_0(x) = R + \frac{1}{Cs+1/x}$ is a simple algebraic iterating function that is completely determined by the Oldham type I fractal chain circuit. Conversely, the iterating function $F_0(x)$ also mathematically fully characterizes its corresponding circuit entity. In this way, by investigating the function $F_0(x)$, it is easy to mathematically uncover the hidden secrets of the circuit. So we have turned analogically a circuit problem into a mathematical problem.

Does this impedance function sequence $\{Z_k(s)\}_{k\in\mathbb{N}}$ converge?

To answer this question, we mathematize the problem. Take $a = R$, $b = 1/(Cs)$, $x_k = Z_k(s)$, and assume they are all positive real numbers, then

$$x_0 \in \mathbb{R}^+ \to x_k = a + \frac{1}{\frac{1}{b} + \frac{1}{x_{k-1}}} = F_0(x_{k-1}), \qquad \begin{pmatrix} a \in \mathbb{R}^+, b \in \mathbb{R}^+ \\ x_k \in \mathbb{R}^+ \end{pmatrix}, \qquad k \in \mathbb{N}^+. \tag{6}$$

If the positive real numerical sequence $\{x_k\}_{k\in\mathbb{N}}$converges, then the impedance function sequence $\{Z_k(s)\}_{k\in\mathbb{N}}$ also converges! Because the iterating function of both is the same function

$$F_0(x) = a + \frac{1}{\frac{1}{b} + \frac{1}{x}}, \begin{pmatrix} a \in \mathbb{R}^+, b \in \mathbb{R}^+, x \in \mathbb{R}^+ \\ \text{Electrical constraint: } a \neq b \end{pmatrix}.$$

3.3. Iterating plane and Convergence: iterating equation and limiting impedance function

Based on iterating formula (6), the iterating planar graph is drawn, as shown in fig. 5. So, we can also use plane geometry to study a circuit problem very intuitively. In particular, the problem of convergence. Because

$$\frac{\mathrm{d}F_0(x)}{\mathrm{d}x} = \left(\frac{b}{b+x}\right)^2 < 1, \begin{pmatrix} x \in \mathbb{R}^+ \\ a \in \mathbb{R}^+, b \in \mathbb{R}^+ \end{pmatrix},$$

according to iteration theory, the corresponding numerical iteration process converges to a fixed point $r_O \in \mathbb{R}^+$, that is, $x_0 \to x_k \xrightarrow{k\to\infty} r_O$. The fixed point r_O can be obtained from the simple iterating equation

$$x = F_O(x) \tag{7}$$

to find

$$r_O = \frac{a}{2} + \frac{a}{2}\sqrt{1 + 4b/a}\,. \tag{8a}$$

Thus, the limit impedance function of the Oldham type Ⅰ prototype circuit is

$$Z_{O1}(s) = \lim_{k\to\infty} Z_k(s) = r_O = \frac{R}{2} + \frac{R}{2}\sqrt{1 + \frac{4}{RCs}}. \tag{8b}$$

So we also call r_O the prototype fixed point, see Fig. 5.

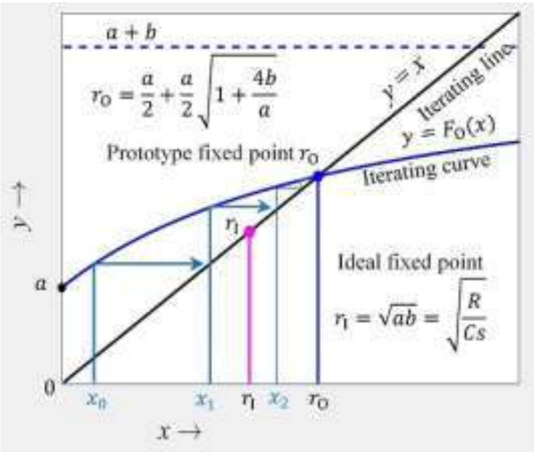


Fig. 5. Oldham Type I iterating curve and fixed points

What does this result mean? All already know this. It was this that has opened up a new field of research for people in the 1960s.

3.4. Operational validity and non-ideal approximation

It has been pointed out above that operational validity is the core problem of the FACs!

Take $\tau = RC$, which is the time constant of the Oldham type Ⅰ fractal chain circuit, corresponding to the characteristic frequency $\Omega_\tau = 1/\tau$. From this, investigating the limit impedance $Z_O(s)$, there is obviously

$$\sqrt{\frac{R}{C}}\, s^{-\frac{1}{2}} \xleftarrow[\text{Low-frequency range}]{0\leftarrow|s|<\Omega_\tau} Z_{O1}(s) \xrightarrow[\text{High-frequency range}]{\Omega_\tau<|s|\to\infty} R\,. \tag{9}$$

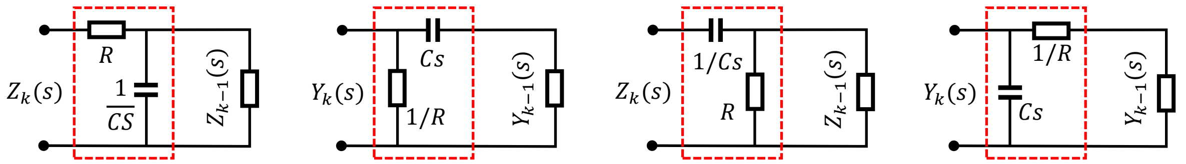
This shows that Oldham type I fractal chain circuit has negative half-order operational performance in a low-frequency range, such a circuit is called a low frequency valid negative half-order fractal FAC. We say that this circuit has Low frequency validity.

Low frequency validity, in contrast, is high frequency validity.

Oldham type III circuit (Fig. 6*(c)* shows its equivalent simplified iteration scheme) is a negative half-order high frequency valid FAC. Its limit impedance $Z_{O3}(s)$ is easy to find and has

$$\frac{1}{Cs} \xleftarrow[\text{Low frequency range}]{0\leftarrow|s|<\Omega_\tau} Z_{03}(s) = \frac{1+\sqrt{1+4RCs}}{2Cs} \xrightarrow[\text{High frequency range}]{\Omega_\tau<|s|\to\infty} \sqrt{R/C}s^{-1/2}. \tag{10}$$

Here we determine the operational validity by solving the simple iterating equation and then investigating the operational performance of the circuit's limit impedance.



Marked by impedance *(a)*Type Ⅰ: LFV

Marked by admittance *(b)* Type II: HFV

Marked by impedance *(c)* Type III: HFV

Marked by admittance *(d)* Type IV: LFV

Fig. 6. Oldham fractal chain circuit class: Equivalent simplified iterating circuits

Of course, there are other ways to determine the operational validity of a given fractal circuit.

Recently, we have proposed a "Brief Analysis Method for fractal circuits" [35] which directly determines the operational validity (or operational performance) of a given prototype fractal circuit based on its topological structure.

Obviously, according to their limiting impedance, Oldham type I and III fractal chain circuits are non-ideal approximation cases. By solving the zeros and poles of the impedance function sequence, it's also easy to verify that both of them are strong approximation cases.

3.5. Mathematical simplification of problems: Normalization and its mathematical description

For the iterating algorithm formula (5), through normalization processing, that is,

$$\frac{Z_k(s)}{R} = 1 + \frac{1}{RCs + \frac{1}{Z_{k-1}(s)/R}} \xrightarrow[\tau=RC,\tau s=w]{\text{Normalizating}} Z_k\left(\frac{w}{\tau}\right)/R = 1 + \frac{1}{RCs + \frac{1}{Z_{k-1}\left(\frac{w}{\tau}\right)/R}}, \tag{11}$$

and let $\tau s = w$, $Z_k\left(\frac{w}{\tau}\right)/R = y_k(w)$(normalized input impedance), we get a normalized iterating algorithm

$$y_0(w) = \frac{N_0(w)}{D_0(w)} \to y_k(w) = F_{01}\big(y_{k-1}(w)\big), \ \ k \in \mathbb{N}^+. \tag{12a}$$

Its normalized iterating function and normalized iterating equation are respectively

$$F_{01}(x) = 1 + \frac{1}{w + 1/x}, \qquad x = F_{01}(x). \tag{12b}$$

Its normalized limit impedance $y_{01}(w)$ (i.e., the normalized prototype fixed point r_{01}) is

$$y_{01}(w) = \lim_{k\to\infty} y_k(w) = \frac{1}{2} + \frac{1}{2}\sqrt{1+4/w} = r_{01}. \tag{13}$$

The normalized operational variable

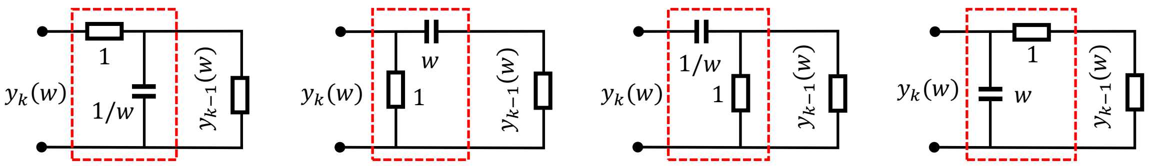
$$w = \tau s = \frac{\sigma}{\Omega_\tau} + \mathrm{j}\frac{\Omega}{\Omega_\tau} = \varsigma + \mathrm{j}\omega. \tag{14}$$

The advantage of normalization treatment is that it makes the study of real physical problems more concise in mathematics, and helps people to more easily to investigate and reveal the essential characteristics of the research object.

Obviously, we have [36] $\dim w = 1$, $[w] = 1 \leftrightarrow [s] = \mathrm{Hz}$, $\dim s = \mathrm{T}^{-1}$; $\dim y_k(w) = 1$, $[y_k(w)] = 1 \leftrightarrow [Z_k(s)] = \Omega$, $\dim Z_k(s) = \mathrm{L}^2\mathrm{MT}^{-3}\mathrm{I}^{-2}$. This is true for all normalized variables!

The impedance of all elements or components in Oldham fractal chain circuits are normalized by the resistance R, and the corresponding normalized iterating circuits are obtained, as shown in fig. 7.

Obviously, these circuit diagrams, compared with fig. 6, fig. 7 are more concise.



Marked by impedance *(a)*Type I: LFV | Marked by admittance *(b)*Type II: HFV | Marked by impedance *(c)*Type III: HFV | Marked by admittance *(d)*Type IV: LFV

Fig. 7. Normalized Oldham fractal chain circuit class:
Equivalent simplified normalized iterating circuits

3.6. Improvement of prototype circuits: Improved circuits and their mathematical descriptions

It is known that Oldham type I fractal chain prototype circuit is a low-frequency valid non-ideal approximation. Consider the normalized case, see fig. 7*(a)* and fig. 8*(a)*, we have (13), that is

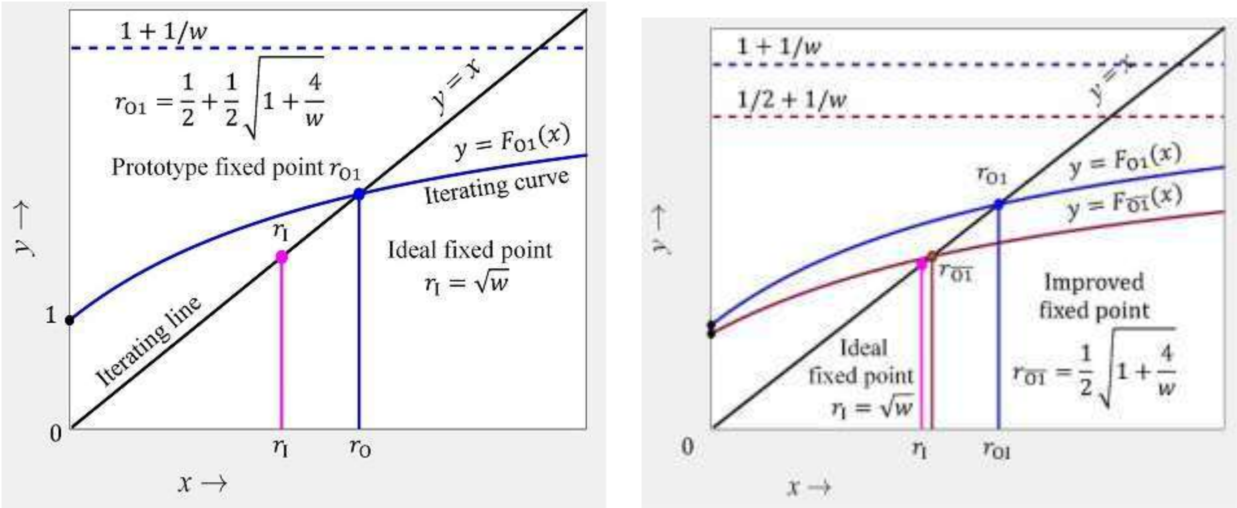
$$y_{01}(w) = \lim_{k\to\infty} y_k(w) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4/w} = r_{01} \neq r_{\mathrm{I}} = \sqrt{1/w}\,. \tag{15}$$

In order to reduce the difference, the fixed point r_{01} and $r_{\mathrm{I}} = \sqrt{1/w}$, the prototype circuit must be modified. The simple improvement measure proposed by Oldham et al [18] is

$$r_{01} - \frac{1}{2} = \frac{1}{2}\sqrt{1 + 4/w} = r_{\overline{01}}\,, \tag{16}$$

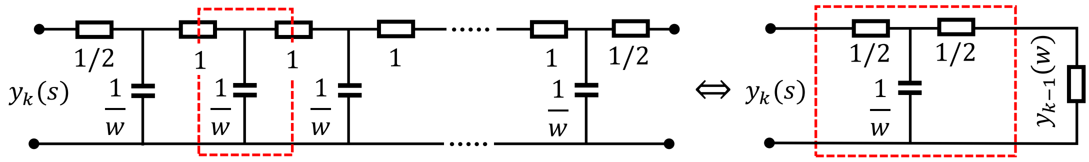
so that the improved fixed point $r_{\overline{01}}$ is more closer to the ideal fixed point $r_{\mathrm{I}} = \sqrt{1/w}$ (see fig. 8*(b)*). Form this, we immediately obtain the improved circuit as shown in fig. 9*(a)*, and its corresponding iterating function and iterating equation, are respectively

$$F_{\overline{01}}(x) = \frac{1}{2} + \cfrac{1}{w + \cfrac{1}{\frac{1}{2} + x}}, \qquad x = F_{\overline{01}}(x)\,. \tag{17}$$

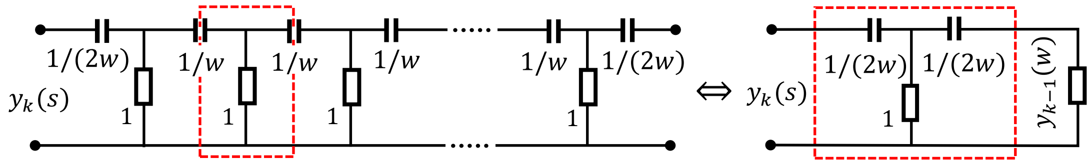


(a) Prototype fixed point and ideal fixed point *(b)* Improved iterating curve and improved fixed point

Fig. 8. Oldham Type I fractal chain circuit—iterating curves and fixed points: Normalized cases



(a) Improved type I and IV: Non-ideal approximation, Low frequency validity



(b) Improved Type II and III: Non-ideal approximation, High frequency validity

Fig. 9. Improved Oldham fractal chain FACs: Normalized cases

Similarly, better approximation performance can be obtained by improving the other three half-order valid Oldham fractal chain circuits (see fig. 6 and fig. 7). Their improved normalized circuits are shown in fig. 9. But they are still negative half-order valid non-ideal approximations. Therefore, a very natural question is: Are there ideal fractance approximation circuits in the full frequency range?

The first full frequency valid ideal approximation FAC was the fractal lattice circuit (see fig. 10), which was proposed by G. E. Carlson in 1960 [14].

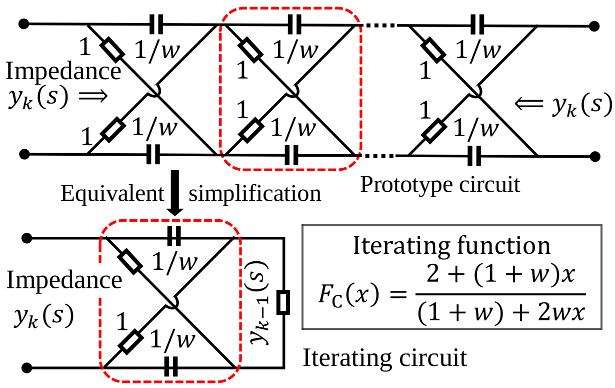


Fig.10. Carlson fractal lattice circuit: Normalized case

3.7. Classical half-order fractal FACs: Ideal approximation case — Full frequency validity

So far, we have obtained 5 kinds of half-order ideal approximation FACs with full-frequency validity. They are the fractal lattice circuit as shown in fig.10 proposed by G. E. Carlson in 1960 [14], the fractal tree circuit as shown in fig.11 proposed by Nakagawa and Sorimach in 1992 [22], the fractal pyramid and fractal tree circuits as shown in fig.12 proposed by Pu Yifei, Yuan Xiao, Liao Ke et al. in 2005 [37-39], and the fractal tree circuit as shown in fig.13 proposed by Yuan Xiao et al in 2013 [1, 40].

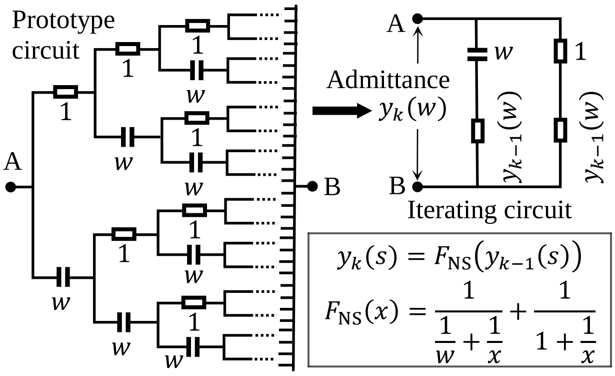


Fig. 11. N-S fractal tree circuit: Normalized case

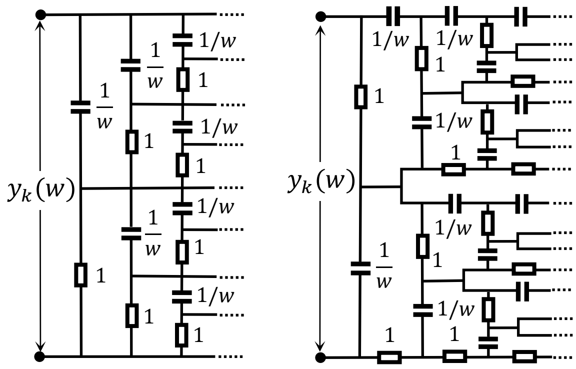


Fig. 12. Pu fractal pyramid and tree circuits: Normalized case

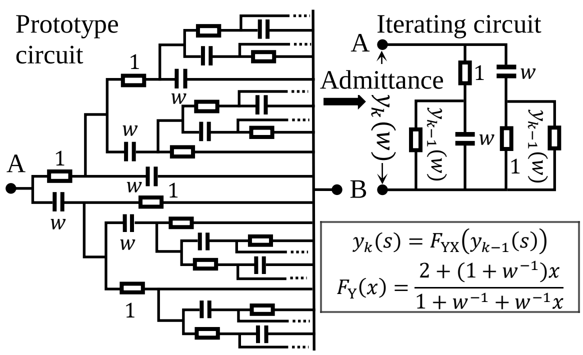


Fig. 13. Yuan fractal tree circuit: Normalized cases

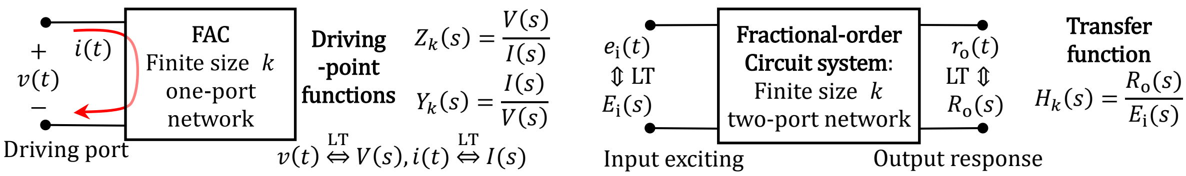
N-S fractal tree circuit and Pu fractal pyramid circuit, Pu fractal tree circuit and Yuan fractal tree circuit are mutually dual circuits. Two FACs that are dual to each other have the same form of iterating function (see section 6.5). All these five fractal circuits can be equivalently reduced to simple iterating circuits, and then the corresponding iterating functions and equations can be obtained. Their normalized limiting impedance or admittance are half-order operator! That is $\lim_{k\to\infty} y_k(w) \propto w^{\pm 1/2}$.

4. Mathematical basis of the frequency-domain analysis: frequency-domain characteristics and operational characteristics

4.1. FACs and fractional-order circuit systems: Mathematical representations

■ **System functions: Driving-point functions and transfer functions**

In general, for a Fractance Approximation Circuit (FAC) with finite size k, it is expressed by the driving-point functions (see fig.14*(a)*): impedance $Z_k(s)$ or admittance $Y_k(s)$; and for a two-port Fractional-Order Circuit with finite size k, it is expressed by the transfer function (see fig. 14*(b)*): $H_k(s)$. These functions are often uniformly called system functions.



(a) One-port Fractance Approximation Circuit *(b)* Two-port Fractional-Order Circuit

Fig. 14. Mathematical representation of simple fractional-order circuit systems

An ideal constant μ-order transfer function is defined as

$$H^{(\mu)}(s) = T^{(\mu)} s^{\mu}, \tag{18}$$

where $T^{(\mu)}$ is a lumped parameter, we call it the transfance [1, 2, 11]. This is the same formally as the ideal μ-order fractance function discussed in section 2.1, see eq. (1): $I^{(\mu)}(s) = F^{(\mu)} s^{\mu}$, where $F^{(\mu)}$ is called the fractance [1, 2, 11, 12].

■ **Normalization treatment and unified mathematical expression**

Without loss of generality, in order to make the description and investigation of problem more concise in mathematics, we can always carry out the following normalization processing and further get a unified mathematical representation:

$$\begin{matrix}\text{System}\\ \text{functions}\end{matrix} \left.\begin{matrix} Z_k(s) \\ Y_k(s) \\ H_k(s) \end{matrix}\right\} \xrightarrow[\tau = RC, \tau s = w]{\text{Normalizating}} \left\{\begin{matrix} Z_k\left(\frac{w}{\tau}\right)/R = \\ RY_k\left(\frac{w}{\tau}\right) = \\ H_k\left(\frac{w}{\tau}\right)/T = \end{matrix}\right\} \underbrace{y_k(w) = \frac{N_k(w)}{D_k(w)}}_{\substack{\text{Normalized} \\ \text{system function}}}; \tag{19a}$$

$$\begin{matrix}\text{Ideal}\\ \text{cases}\end{matrix} \left.\begin{matrix} I^{(\mu)}(s) = F^{(\mu)} s^{\mu} \\ H^{(\mu)}(s) = T^{(\mu)} s^{\mu} \end{matrix}\right\} \xrightarrow[\tau s = w]{\text{Normalizating}} \left\{\begin{matrix} \dfrac{I^{(\mu)}\left(\frac{w}{\tau}\right)}{F^{(\mu)}/\tau^{\mu}} = \\ \dfrac{H^{(\mu)}\left(\frac{w}{\tau}\right)}{T^{(\mu)}/\tau^{\mu}} = \end{matrix}\right\} \iota^{(\mu)}(w) = w^{\mu}; \tag{19b}$$

$$\begin{matrix}\text{Non}-\text{ideal}\\ \text{cases}\end{matrix} \left.\begin{matrix} \tilde{I}^{(\mu)}(s) \\ \tilde{H}^{(\mu)}(s) \end{matrix}\right\} \xrightarrow[\tau s = w]{\text{Normalizating}} \tilde{\iota}^{(\mu)}(w) \underbrace{\approx \iota^{(\mu)}(w) = w^{\mu}}_{\text{Additional conditions}}. \tag{19c}$$

In Section 3.5, normalizing the Oldham fractal chain circuits is a successful example.

4.2. Frequency response — Frequency-domain characteristics: Order-frequency characteristics

When studying the signal analysis and processing ability of an analog circuit system, it is usually carried out in the frequency domain.

■ **Frequency response: Amplitude-frequency characteristics and phase-frequency characteristics**

In the normalized system functions $y_k(w)$ and $\iota^{(\mu)}(w)$, etc, take $w = \mathrm{j}\omega$, we get the frequence response:

$$y_k(\mathrm{j}\omega) = A_k(\omega) \mathrm{e}^{\mathrm{j}P_k(\omega)} \Leftrightarrow \begin{cases} A_k(\omega) = |y_k(\mathrm{j}\omega)| \\ P_k(\omega) = \arg\{y_k(\mathrm{j}\omega)\} \end{cases} \left(\begin{matrix} \mathrm{j} = \sqrt{-1} \\ \omega \in \mathbb{R} \end{matrix}\right), \tag{20a}$$

$$\iota^{(\mu)}(\mathrm{j}\omega) = A^{(\mu)}(\omega)\mathrm{e}^{\mathrm{j}P^{(\mu)}(\omega)} \Leftrightarrow \begin{cases} A^{(\mu)}(\omega) = \left|\iota^{(\mu)}(\mathrm{j}\omega)\right| = |\omega|^{\mu} \\ P^{(\mu)}(\omega) = \arg\left\{\iota^{(\mu)}(\mathrm{j}\omega)\right\} = \frac{\pi}{2}\mu \operatorname{sig}\omega \end{cases} \quad \begin{pmatrix} \mathrm{j} = \sqrt{-1} \\ \omega \in \mathbb{R} \end{pmatrix}. \quad (20\mathrm{b})$$

Where $A_k(\omega)$ and $A^{(\mu)}(\omega)$ are called amplitude-frequency (or magnitude-frequency) characteristic functions, referred to as amplitude-frequency characteristics; $P_k(\omega)$ and $P^{(\mu)}(\omega)$ are called phase-frequency characteristic functions or phase-frequency characteristics for short.

■ **Bode representation of the frequency response**

In the field of engineering technology, Bode expressions and Bode curves are usually used to better investigate and study the intrinsic properties of the system. That is, taking

$$\omega = 10^{\varpi} \Leftrightarrow \varpi = \lg\omega\,, \qquad \varpi \in \mathbb{R}, \omega \in \mathbb{R}^{+},$$

one get the amplitude-frequency characteristic functions in the form of double logarithmic coordinates

$$\Lambda_k(\varpi) = \lg A_k(10^{\varpi})\,,\ \Lambda^{(\mu)}(\varpi) = \lg A^{(\mu)}(10^{\varpi}) = \mu\varpi,\ \ \varpi \in \mathbb{R}\ \ , \qquad (21)$$

we still call they the amplitude-frequency characteristics or magnitude-frequency characteristics; and the phase-frequency characteristic functions in the form of single logarithmic coordinates

$$\theta_k(\varpi) = P_k(10^{\varpi}), \qquad \underbrace{\Theta^{(\mu)}(\varpi) = P^{(\mu)}(10^{\varpi}) = \frac{\pi\mu}{2}}_{\text{Constant Phase Property}}, \qquad \varpi \in \mathbb{R}\ \ , \qquad (22)$$

we still call they the phase-frequency characteristics.

■ **Frequency response: Order-frequency characteristics**

For fractional-order circuits and systems, especially, for FACs, the operational characteristics are very important. About 15 years ago, after many considerations and experimental verification, we proposed a new frequency-domain characteristic functions—order-frequency characteristic functions or order-frequency characteristics for short, which is defined as [1, 2]

$$\mu_k(\varpi) = \frac{\mathrm{d}\Lambda_k(\varpi)}{\mathrm{d}\varpi}, \qquad \underbrace{M^{(\mu)}(\varpi) = \frac{\mathrm{d}\Lambda^{(\mu)}(\varpi)}{\mathrm{d}\varpi} = \mu}_{\text{Constant Order Property}}, \qquad (\varpi \in \mathbb{R}). \qquad (23)$$

They are used to describe the calculus operational capability and operational features of analog circuits and systems, especially when emphasizing the analysis of operational performances.

Constant phase and constant order in the frequency-domain are the most essential properties of constant order fractance approximation circuits or fractional-order circuits and systems!

Order-frequency and phase-frequency characteristics, we unified called operational characteristics.

4.3. Operational characteristics of first-degree metasystem

The normalized system function of a first-degree metasystem with negative real zero-pole pair $(z_i,\ p_i)$ is defined as [1, 2]

$$m_i(w) = \frac{w - z_i}{w - p_i} = \frac{w + 10^{o_i}}{w + 10^{\chi_i}} \quad \begin{pmatrix} o_i \in \mathbb{R} \\ \chi_i \in \mathbb{R} \end{pmatrix}, \qquad (24)$$

its phase-frequency characteristics $\vartheta_i(\varpi)$ and order-frequency characteristics $u_i(\varpi)$ are respectively

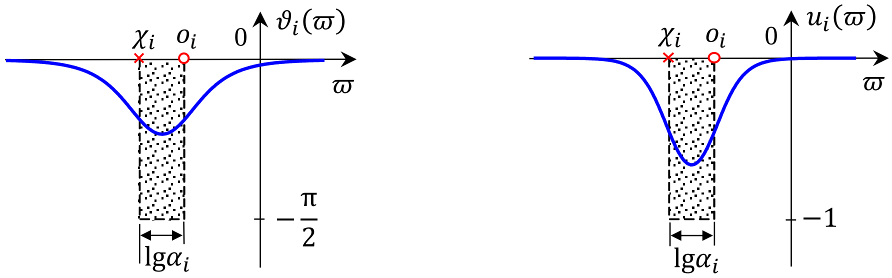
$$\vartheta_i(\varpi) = \arctan\frac{\sinh\left(-\frac{1}{2}\ln\alpha_i\right)}{\cosh\left((\varpi-\varpi_i)\cdot\ln 10\right)} \quad \begin{pmatrix}\alpha_i = 10^{o_i-\chi_i} = z_i/p_i \\ \varpi_i = (o_i+\chi_i)/2\end{pmatrix}, \tag{25a}$$

$$u_i(\varpi) = \arctan\frac{\sinh(\ln\alpha_i)}{\cosh(2(\varpi-\varpi_i)\cdot\ln 10)+\cosh(\ln\alpha_i)}. \tag{25b}$$

Both of them have highly localization features, as shown in figure 14, that is, we have

$$\int_{\varpi\in\mathbb{R}} \vartheta_i(\varpi)\mathrm{d}\varpi = \frac{\pi}{2}\lg\alpha_i, \quad \int_{\varpi\in\mathbb{R}} u_i(\varpi)\mathrm{d}\varpi = \lg\alpha_i, \quad \left(\alpha_i = \frac{z_i}{p_i}\right). \tag{26}$$

Where $\alpha_i = 10^{o_i-\chi_i} = z_i/p_i$ is zero-pole ratio. These conclusions are crucial to our research of FACs!



(a) Phase-frequency characteristics *(b)* Order-frequency characteristics

Fig. 14. Operational characteristic curves of the first-degree metasystem

It is these localization features that provide a theoretical support for us to define the order-frequency characteristic functions as (23), and at the same time, it is the mathematical theoretical basis for us to understand and develop the FACs, and many valid rational approximation methods of fractional operators!

4.4. Operational characteristics of Oldham fractal chain circuits: Non-ideal approximation case

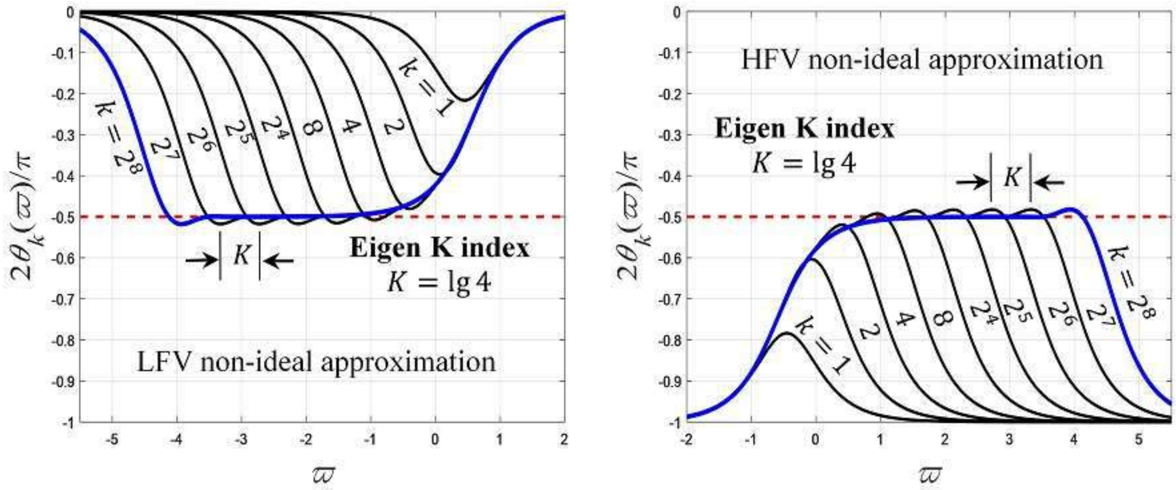
Consider the normalized Oldham improved type I and II fractal chain circuits. Its input impedance sequence can be obtained and expressed analytically as [1, 2, 41]

$$y_0(w) = \infty \to y_{\bar{\mathrm{I}}k}(w) = \frac{1}{2}\sqrt{1+4/w}\,\coth\left(k\cdot\operatorname{acosh}\left(1+\frac{w}{2}\right)\right) \xrightarrow[\textbf{LFV}]{\substack{k\to\infty \\ 1>|w|\to 0}} \sqrt{1/w}, \tag{27a}$$

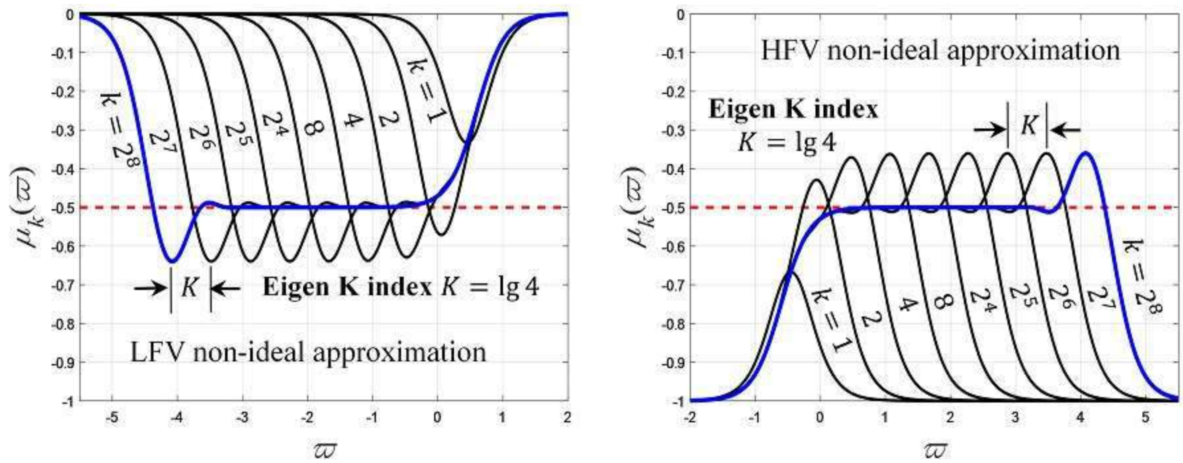
$$y_0(w) = \infty \to y_{\bar{\mathrm{II}}k}(w) = \frac{1}{2w}\sqrt{1+4w}\,\tanh\left(k\cdot\operatorname{acosh}\left(1+\frac{1}{2w}\right)\right) \xrightarrow[\textbf{HFV}]{\substack{k\to\infty \\ 1<|w|\to \infty}} \sqrt{1/w}. \tag{27b}$$

Of course, there are several effective algorithms for solving finite length $\{y_k(w)\}$, especially numerical methods. In the field of FACs, calculating $\{y_k(w)\}$ is a very important and necessary basic task. Accordingly, their characteristic curves in the frequency-domain are plotted, as shown in fig. 15. In these graphs, the red dash straight lines corresponds to the case of an ideal negative half-order fractor, that is $w^{-1/2}$.

The $K = \lg 4$ in fig.15, we call the eigen K index or K index for short. The eigen K index profoundly characterizes the approximation performance and iterating convergence performance of the corresponding circuit [1, 2].



(*a*) Phase-frequency characteristic curves: Improved type I, improved type II



(b) Order-frequency characteristic curves: Improved type I, improved type II

Fig. 15. Operational characteristic curves: Normalized Improved Oldham type I and II

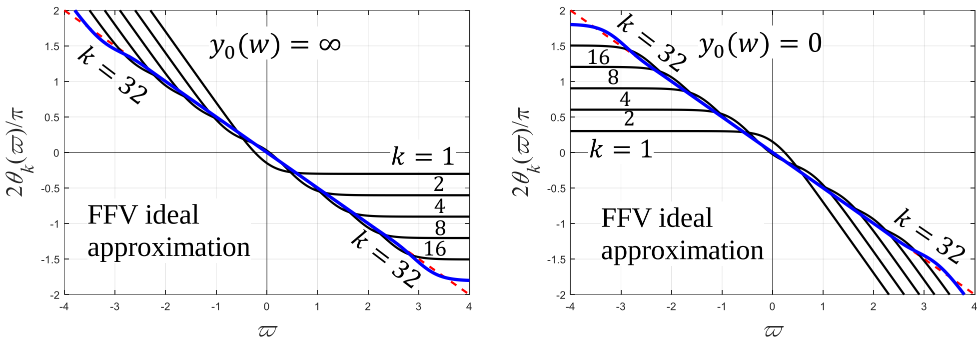
4.5. Operational characteristics of Carlson fractal lattice circuits: FFV ideal approximation

Consider the normalized Carlson fractal lattice circuit. Its input impedance sequences can be obtained and expressed analytically as [1, 2, 41]

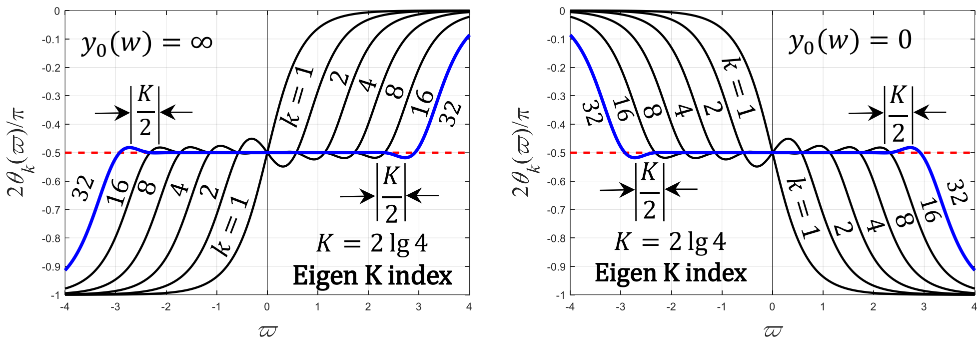
$$\begin{cases} y_0(w) = \infty \to y_{\mathrm{O}k}(w) = \dfrac{1}{\sqrt{w}}\coth\left(k \cdot \operatorname{acosh}\left(\dfrac{w+1}{w-1}\right)\right) \xrightarrow[\mathbf{FFV}]{k\to\infty} \dfrac{1}{\sqrt{w}}, \\ y_0(w) = 0 \to y_{\mathrm{S}k}(w) = \dfrac{1}{\sqrt{w}}\tanh\left(k \cdot \operatorname{acosh}\left(\dfrac{w+1}{w-1}\right)\right) \xrightarrow[\mathbf{FFV}]{k\to\infty} \dfrac{1}{\sqrt{w}}. \end{cases} \tag{28}$$

Thus, the frequency-domain characteristic curves is drawn, as shown in fig. 16. In this FFV ideal approximation case, the eigen K index $K = 2\lg 4$.

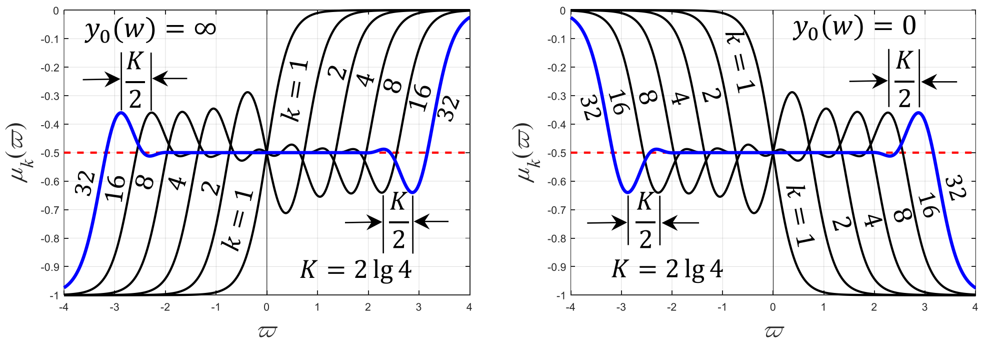
Why are there such conclusions for the eigen K index of these half-order valid FACs? Can anyone prove these mathematically and rigorously? As far as the author knows, this is a problem that has not been solved yet! However, using the brief analysis method of fractal circuit, it can be roughly obtained in a non-strict sense that the eigen K index are indeed so [35].



(a) Amplitude-frequency characteristic curves



(b) Phase-frequency characteristic curves



(c) Order-frequency characteristic curves

Fig. 16. Frequency-domain characteristic curves: Normalized Carlson fractal lattice circuit

5. Liu-Kaplan fractal chain circuits and their mathematical descriptions

There is a hidden order everywhere, and mathematics can sometimes reveal it.

5.1. Fractal model for the ac response of a rough interface

In 1985, for a rough interface between two materials of very different conductivities, e.g., an electrode and an electrolyte, based on the morphological-geometric characteristics, S. H. Liu cleverly and ingeniously proposed a fractal model, namely the regular Cantor fractal-bar model as shown in figure 17*(a)*. The analogical equivalent circuit of this fractal model (see figure 17*(b)*), which takes into consideration the resistance in the two substances and the capacitance of the interface, has the property of the so-called constant-phase element, i.e., a passive circuit element whose complex impedance has a power-law singularity at low frequencies. The exponent of the frequency dependence is related to the fractal dimension. The model also provides insight into the conducting properties of the percolating cluster and the source of the 1/f noise in electronic components [19].

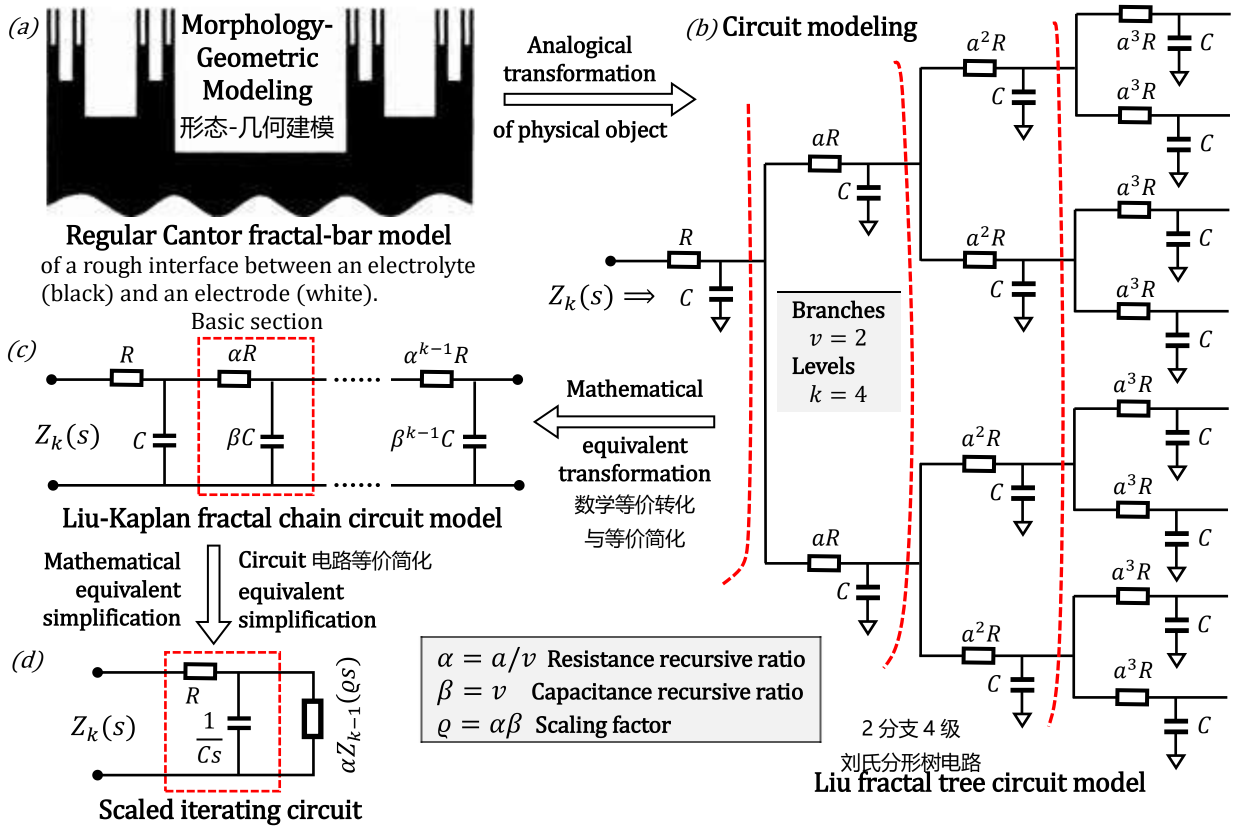


Fig. 17. Analogical transformation and equivalent simplification: Liu fractal model→Liu-Kaplan fractal chain circuits: *(a)* Regular Cantor fractal-bar model Morphology-geometric modeling; *(b)* Circuit modeling —Liu fractal tree circuit; *(c)* Liu-Kaplan fractal chain circuit; *(d)* Scaled iterating circuit

To put it simply, the analogical equivalent circuit has the ability to realize arbitrary-order fractional operators in the low frequency range by adjusting the element parameters and structure parameters. In order to deeply express our respect for Liu, we called this circuit as the Liu's fractal tree circuit [1, 2].

5.2. Analogical transformation and equivalent simplification of problems and their mathematical description—Irregular Liu-Kaplan scaling equation

■ Analogical transformation and circuit modeling

By analogical transformation of physical objects, the equivalent modeling of the regular Cantor fractal-bar interface into Liu's fractal tree is a crucial first step. Its input impedance function $Z_k(s)$, according to the circuit structure characteristics, can be directly written in finite irregular continued fraction form,

$$Z_k(s) = R + \frac{1}{Cs +}\frac{v}{aR +}\frac{1}{Cs +}\frac{v}{a^2R +}\frac{1}{Cs +}\frac{v}{a^3R +}\frac{1}{Cs +}\ddots. \tag{29}$$

■ Mathematical equivalent transformation — Regular continued fraction and Liu-Kaplan fractal chain circuit

Equation (29) can be equivalently transformed to regular (or simple) continued fraction form,

$$Z_k(s) = R + \frac{1}{Cs +}\frac{1}{\alpha R +}\frac{1}{\beta Cs +}\frac{1}{\alpha^2 R +}\frac{1}{\beta^2 Cs +}\ddots, \qquad \begin{pmatrix} \alpha = a/v \in \mathbb{R}^+ \\ \beta = v \in \mathbb{R}^+ \end{pmatrix}. \tag{30}$$

From this, we obtain the Liu-Kaplan fractal chain circuit shown in fig. 17*(c)*. Where, $\alpha = a/v$, we call the resistance progression ratio, and $\beta = v$, we call the capacitance progression ratio, they are collectively referred to as the scaling feature parameters [1, 2].

■ Mathematical equivalent simplification — Scaled iterating formula and scaled iterating circuit

Further, Eq. (30) is simplified equivalently to scaled iterating form,

$$Z_0(s) = \frac{N_0(s)}{D_0(s)} \to Z_k(s) = R + \cfrac{1}{Cs + \cfrac{1}{\alpha Z_{k-1}(\varrho s)}}, \qquad \begin{pmatrix} k \in \mathbb{N}^+ \\ \varrho = \alpha\beta \in \mathbb{R}^+ \end{pmatrix}. \tag{31}$$

Where $\varrho = \alpha\beta \neq 1$, we call it the scaling factor. From this we draw the scaled iterating circuit shown in fig. 17*(d)* [1, 2, 18-20]!

The three circuits in fig.17, namely the Liu fractal tree circuit, the Liu-Kaplan fractal chain circuit and the scaled iterating circuit, are functionally equivalent!

5.3 Determination of the operational validity—Liu-Kaplan scaling equation and Liu's rough solution

■ Limiting impedance and Liu-Kaplan scaling equation

The input impedance sequence $\{Z_k(s)\}_{k\in\mathbb{N}}$, which iterated from (31), if convergent, then its limiting impedance $Z_{\text{LK}}(s) = \lim_{k\to\infty} Z_k(s)$ is found by the irregular scaling equation—**Liu-Kaplan scaling equation**

$$Z(s) = R + \cfrac{1}{Cs + \cfrac{1}{\alpha Z(\varrho s)}}, \quad \begin{pmatrix} \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}^+ \\ \varrho = \alpha\beta \in \mathbb{R}^+ \end{pmatrix}. \tag{32}$$

We call this Liu-Kaplan scaling equation because it was first exactly derived by T. Kaplan et al. in 1985 [18-20]. So far, as far as the author knows, this is an irregular scaling equation that can not be also solved analytically, or it is extremely difficult to solve analytically. Perhaps it is the author's solitary omissions, only shallow knowledge, sincerely seek the master to learn.

■ Regular scaling equation and approach analytical solution —Liu's rough solution

Form the circuit point of view, as in section 3.4 for the Oldham fractal chain circuits, see expressions (9), (10), we have (let $\Omega_\tau = \frac{1}{RC}$)

$$\alpha Z(\varrho s) \approx Z(s) \xleftarrow[\text{Low frequency}]{0\leftarrow|s|<\Omega_\tau} Z(s) = R + \cfrac{1}{Cs + \cfrac{1}{\alpha Z(\varrho s)}} \xrightarrow[\text{High frequency}]{\Omega_\tau<|s|\to\infty} Z(s) \approx R. \tag{33}$$

This leads to a regular scaling equation in the low frequency range: $Z(s) \approx \alpha Z(\varrho s)$. Thus, an approach analytical solution— Liu's approach solution is obtained [18] :

$$Z_{\text{LK}}(s) \approx Z_{\text{Liu}}(s) = \kappa s^{\mu_{\text{Liu}}}, \qquad \mu_{\text{Liu}} = -\frac{\lg\alpha}{\lg\varrho}. \tag{34}$$

Where κ is a scalar factor; μ_{Liu}, we call it Liu's operational order, or Liu's order for short; The power function $Z_{\text{Liu}}(s) = \kappa s^{\mu_{\text{Liu}}}$ is called Liu's rough solution [1, 2] .

According to the operational characteristics of Liu's rough solution, the operational validity of the circuits described by the irregular scaling equation can be preliminarily determined. These circuits are low frequency valid FACs, and more importantly, by adjusting the scaling feature parameters, i.e. α and β, one can get arbitary negative fractional-order FACs in the low frequency range!

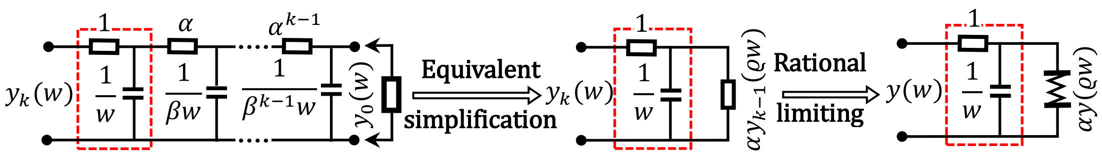
The Oldham fractal chain circuits have only negative half-order operational ability.

5.4 Normalization processing and the frequency-domain characteristics

■ Normalization: Scaled iterating circuit and iterating function

Let $\tau = RC$, $w = \tau s$, the Liu-Kaplan fractal chain circuits (Fig. 17 *(c)*, *(d)*) are normalized, and the results are shown in fig. 18. The corresponding normalized iterating algorithm formula is

$$y_0(w) = \frac{N_0(w)}{D_0(w)} \to y_k(w) = 1 + \frac{1}{w + \frac{1}{\alpha y_{k-1}(\varrho w)}}, \quad k \in \mathbb{N}^+, \tag{35}$$



(a) Prototype circuit *(b)* Scaled iterating circuit *(c)* Irregular scaling equation circuit

Fig. 18. Normalized Liu-Kaplan fractal chain circuits

and the normalized Liu-Kaplan scaling equation is

$$y(w) = F_{\mathrm{OI}}\big(\alpha y(\varrho w)\big), \quad F_{\mathrm{O1}}(x) = 1 + \frac{1}{w + \frac{1}{x}}. \tag{36}$$

This is an irregular scaling equation, and its corresponding circuit representation is shown in fig.18*(c)*, which is called an irregular scaling equation circuit. Here, its iterating function is exactly the iterating function $F_{\mathrm{O1}}(x)$ that describes the Oldham (type I) fractal chain circuit! See expressions (6) and (12).

$F_{\mathrm{O1}}(x)$, this seemingly extremely simple function, is the key to unclocking our understanding of the circuits it describes and their complex physical systems and processes.

■ Frequency-domain characteristics: Negative half-order case and operational oscillating effects

According to the algorithm formula (35), in MATLAB , it is easy to program, and to solve numerically the finite-length input impedance function sequence $\{y_k(w)\}_{k=1\sim K}$. Thus, from this, the frequency-domain characteristic curves are drawn for investigation and analysis.

Let's first consider the case of Liu's order

$$\mu_{\mathrm{Liu}} = -1/2, \text{ i.e. } \alpha = \beta = \sqrt{\varrho} > 1. \tag{37}$$

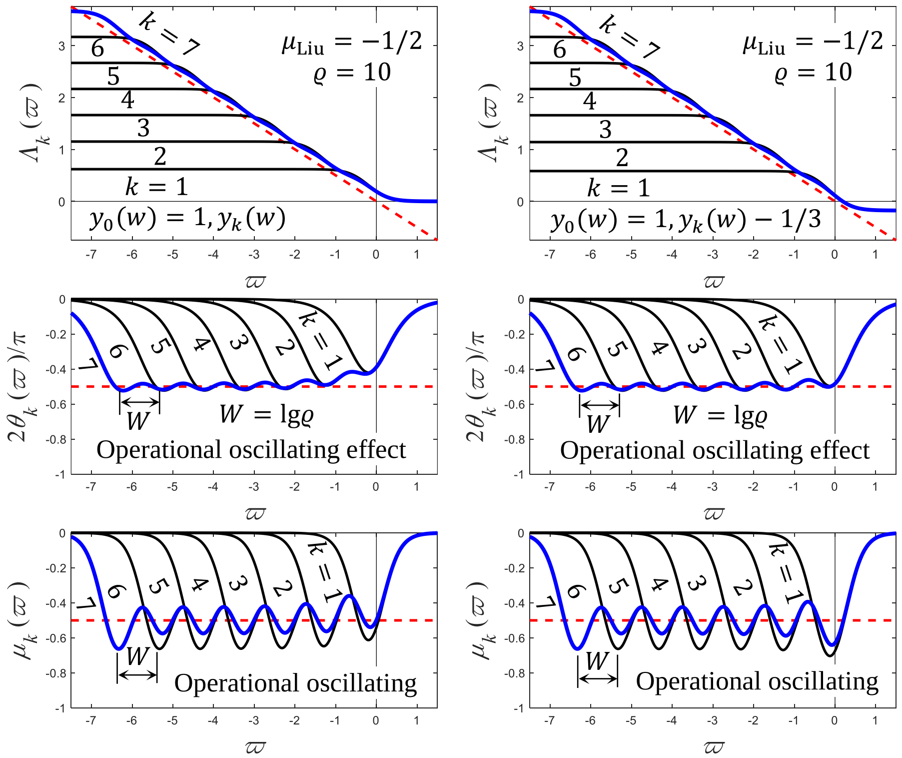
Some of the results are shown in fig.19.

By observing a large number of numerical experiment results, the following qualitative conclusions are preliminary listed:

1) The value of the initial impedance $y_0(w)$, affects the operational performance and approximation performance in the very low frequency range.
2) A simple change in the circuit will improve the approximation performance and thus improve the operational performance. This is similar to the Oldham fractal chain circuits.
3) In contrast to the Oldham fractal chain circuit, the Liu-Kaplan fractal chain circuit has a deterministic periodic operational oscillating effect in the approximation frequency band (see

the phase frequency and order frequency characteristic curves in fig. 19)! Its oscillating period $W=\lg\varrho$, where $\varrho=\alpha\beta>1$, is the scaling factor.

4) Here, the operational oscillating effects, break the constant phase property and constant order property which is expected by the FACs!



(a) Prototype circuit: $y_k(w)$ *(b)* Improved circuit: $y_k(w)-1/3$

Fig.19. Frequency-domain characteristics: $\mu_{\mathrm{Liu}}=-1/2$, $\varrho=10$, $y_0(w)=1$

Such periodic oscillation fluctuates in the approximation frequency-band around the invariant characteristic horizontal line (i.e., red dashed lines in fig.19) of the ideal operator at Liu's order, i.e. $\kappa w^{\mu_{\mathrm{Liu}}}$, $\mu_{\mathrm{Liu}}=-1/2$. Why is this?

■ **Frequency-domain characteristics: General Negative fractional-order cases and operational oscillating effects**

Consider the case of Liu's order $\mu_{\mathrm{Liu}}=-j/10$ ($j=1,3,5,7,9$) when scaling factor ϱ is given. The frequency-domain characteristic curves are drawn by $y_k(w)$, as shown in fig. 20, 21.

For scaling fractance approximation circuits, in general, there is always an inherent quasi-periodic operational oscillating effect [1]. Its oscillating period $W=\lg\varrho$. This is because for an irregular scaling equation, such as Liu-Kaplan scaling equation (36), under logarithmic scale, we have

$$y(\mathrm{j}\cdot 10^{\varpi})=F_{01}\left(\alpha y\left(\mathrm{j}\cdot 10^{\varpi+\lg\varrho}\right)\right),\ \ y(\mathrm{j}\cdot 10^{\varpi})=1+\cfrac{1}{w+\cfrac{1}{\alpha y(\mathrm{j}\cdot 10^{\varpi+\lg\varrho})}}. \tag{38}$$

Therefore, the so-called quasi-periodic phenomenon here is also called log-periodicity in some literature [8]. In the approximation frequency band, we have [1]

$$\theta_k(\varpi) \approx \frac{\pi}{2}\mu_{\mathrm{Liu}} + m_{\mathrm{o}}(\mu_{\mathrm{Liu}}, \varrho)\sin\left(\frac{2\pi}{\lg\varrho}\varpi + \vartheta(\mu_{\mathrm{Liu}}, \varrho)\right), \tag{39a}$$

$$\mu_k(\varpi) \approx \mu_{\mathrm{Liu}} + m_{\mathrm{p}}(\mu_{\mathrm{Liu}}, \varrho)\sin\left(\frac{2\pi}{\lg\varrho}\varpi + \vartheta(\mu_{\mathrm{Liu}}, \varrho)\right). \tag{39b}$$

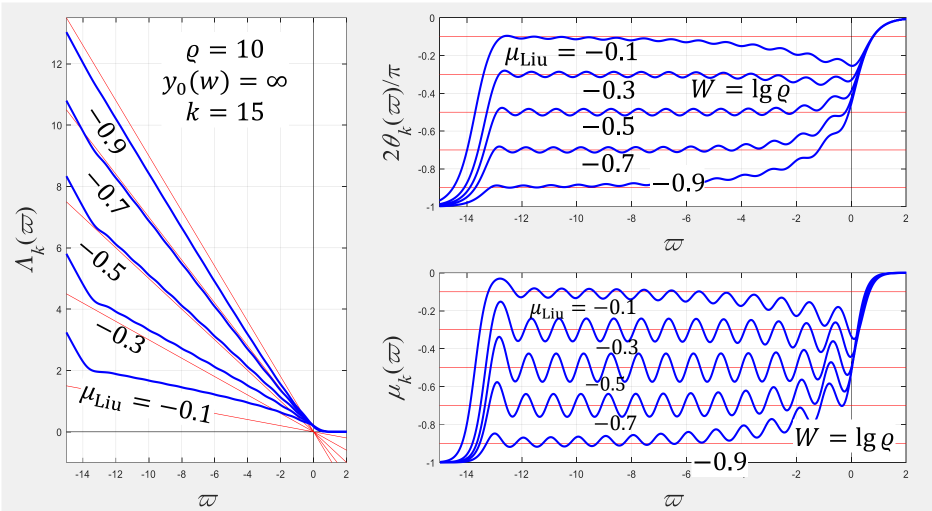


Fig. 20. Frequency-domain characteristics: $\varrho = 10,\ y_0(w) = \infty,\ k = 15$

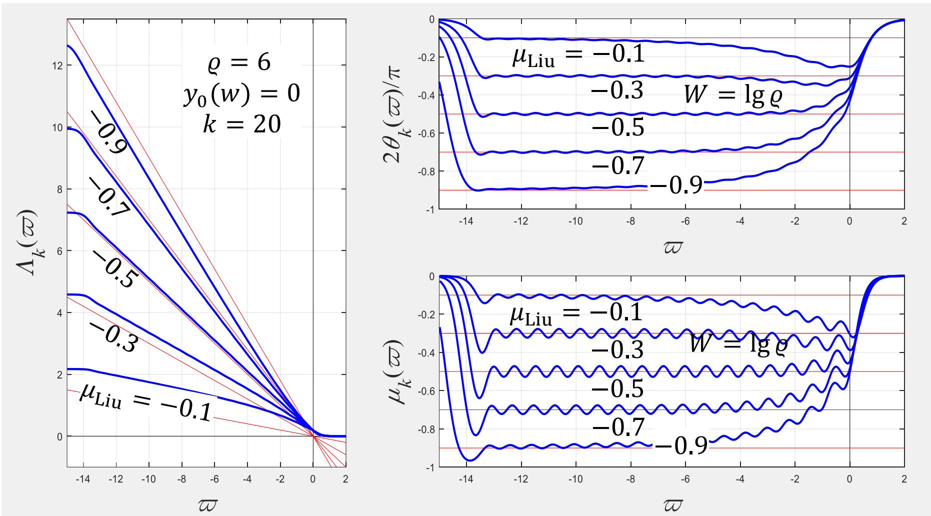


Fig. 21. Frequency-domain characteristics: $\varrho = 6,\ y_0(w) = 1,\ k = 20$

6. Scaling extension theory and irregular scaling equations

The design and fabrication of arbitrary real-order μ ($0 < |\mu| < 1$) fractance approximation circuits (FACs) is the lofty goal of physical realization and application of ideal fractional operator, or ideal fractance function

$$I^{(\mu)}(s) = F^{(\mu)}s^{\mu} \xrightarrow[\text{Normalizating}]{\tau s = w} \iota^{(\mu)}(w) = w^{\mu} \quad \begin{pmatrix} 0 < |\mu| < 1 \\ \mu \in \mathbb{R} \end{pmatrix}. \tag{40}$$

In the absence of fractional-order components, people can only try their best to achieve and synthesize (approximately) the operational performance of fractional operators by means of existing elements, devices and technologies. Or observe and analyze a variety of (inorganic and organic) materials, devices (or biological organs, tissues), complex real systems and their behaviors, as well as a large number of fractional phenomena and processes occurring in physics (especially nanophysics), chemistry, biology, medicine, engineering mechanics and other fields, even if only within a certain frequency range, it is beneficial to establish the model of FACs.

6.1. Traditional methods of designing and constructing FACs

Historically, there have been two main paths to research and development of FACs.

■**Analogical circuit modeling**

In scientific and experimental research, FACs are modeled by use of analogical transformation, equivalent simplification, and other means based on a large number of complex real-world systems with fractional-order processes and phenomena. We call this method “the analogical circuit modeling” [1, 2]. For example,

★homogeneous distributed RC networks,

★Oldham negative half-order fractal chain circuits [18],

★Liu’s fractal tree circuits [19-21],

★Nakagawa-Sorimach fractal tree [22],

★Haba fractional capacitors [23-26],

etc., are very convincing typical results in the analogical circuit modeling.

■**Mathematical rational approximation**

Based on a variety of mathematical techniques, within a certain frequency range, the physical realizable rational approximation function sequence of the fractional operator is first theoretically carried out, and then transformed into a practical circuit (especially passive circuit networks) [1]. For example,

★Carlson half-order fractal lattice circuit [14],

★Carlson $\pm 1/n$-order regular iterating algorithm [15],

★Dutta Roy continued fraction expansion [16, 17],

★Charef arbitrary order method [42-45],

★Matsuda log-frequency point CFE method [46],

★Oustaloup zero-pole construction method [47],

and so on are the models of mathematical rational approximation.

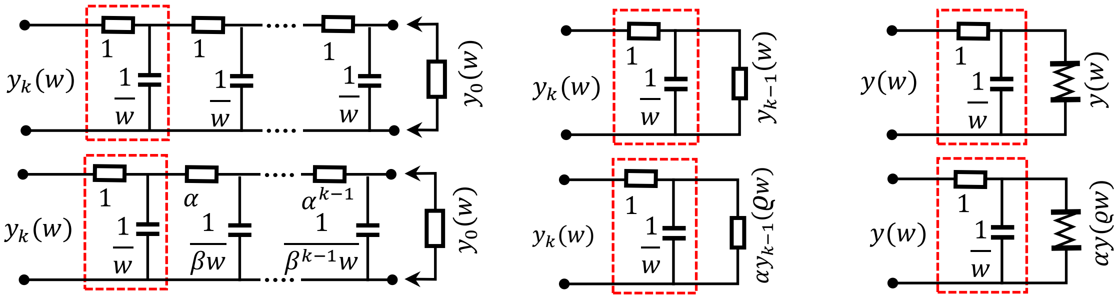
Is there a third way to building new type of FACs? In particular, new arbitrary-order valid FACs!

6.2 Scaling extension theory and its mathematical descriptions

Comparing **Low Frequency Valid** half-order Oldham type Ⅰ and arbitrary-order Liu-Kaplan fractal chain circuits shown in fig. 22, and the beautiful iterating equations describing them (normalized cases):

Oldham: **Half-order valid** Liu-Kaplan: **Arbitrary order** Shared iterating function

$$\underbrace{y(w) = F_{01}\big(y(w)\big),}_{\text{Algebraic iterating equation}} \qquad \underbrace{y(w) = F_{01}\big(\alpha y(\varrho w)\big),}_{\text{Irregular scaling equation}} \quad F_{01}(x) = 1 + \cfrac{1}{w + \cfrac{1}{x}}.$$



(a) Prototype circuits *(b)* Simple iterating circuits *(c)* Iterating equation circuits

Fig. 22. Normalized Oldham type I and Liu-Kaplan fractal chain circuits:
$1 < \alpha < \infty,\ 1 < \beta < \infty,\ \varrho = \alpha\beta$.

■Scaling extension: Basic concepts and their mathematical descriptions

According to the above comparative investigation. it is very easy to see that Oldham type Ⅰ is a special case of Liu-Kaplan fractal chain circuits (i.e., $\alpha = \beta = \varrho = 1$), and Liu-Kaplan fractal chain circuit is a generalization of Oldham type Ⅰ, we call this scaling extension [48, 49] ! The corresponding mathematical statement is that the half-order valid algebraic iterating equation is scaled to become (possibly) an arbitrary-order valid irregular scaling equation (considering the normalization case):

$$\underbrace{y(w) = F\big(y(w)\big)}_{\substack{\text{half-order valid}\\ \text{iterating equation}}} \xrightarrow[\substack{0<\alpha<\infty,\ \alpha\neq 1\\ 0<\beta<\infty,\ \beta\neq 1\\ \varrho=\alpha\beta\neq 1}]{\text{Scaling extension}} \underbrace{y(w) = F\big(\alpha^{\pm 1} y(\varrho^{\pm 1} w)\big)}_{\substack{\text{arbitary-order valid}\\ \text{irregular scaling equation}}}. \tag{41}$$

Where α and β are still called scaling feature parameters, and $\varrho = \alpha\beta \neq 1$ is scaling factor.

■ Low-frequency validity and direct proportion extension

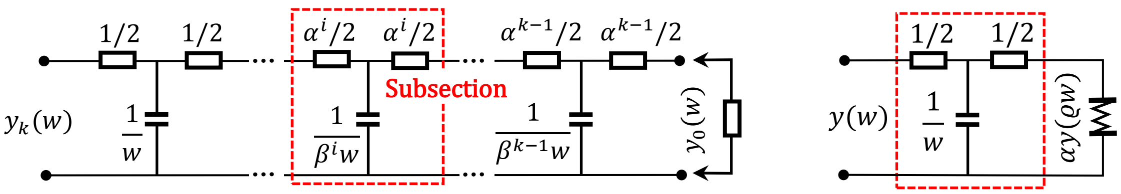
In fig. 22, for the half-order low frequency valid (LFV) FAC, after scaling extension, it is still a low frequency valid arbitrary FAC, but the following conditions must be met:

$$1 < \alpha < \infty,\ 1 < \beta < \infty,\ 1 < \varrho = \alpha\beta < \infty. \tag{42}$$

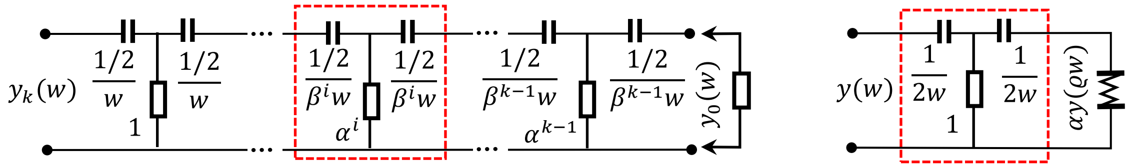
Scaling extension that satisfies condition (41) is called **direct proportion extension** (DPE).

By direct proportion extension of the improved Oldham fractal chain circuit (see fig.9*(a)*, and expression (17)), which is a half-order low frequency valid (LFV) circuit, we obtain a new scaled fractal chain circuit as shown in fig.23*(a)*, which is still a LFV, but has an arbitrary operational order, namely Liu's order $\mu_{\text{Liu}} = -\lg\alpha / \lg\varrho$. The scaling equation describing this scaled circuit is irregular, i.e.

$$y(w) = F_{\overline{01}}\big(\alpha y(\varrho w)\big) = \frac{1}{2} + \frac{1}{w + \frac{1}{\frac{1}{2} + \alpha y(\varrho w)}}. \tag{6.4}$$



(a) LFV → direct proportion extension → LFV: $1 < \alpha < \infty,\ 1 < \beta < \infty,\ \varrho = \alpha\beta$



(b) HFV → inverse proportion extension → HFV: $0 < \alpha < 1,\ 0 < \beta < 1,\ \varrho = \alpha\beta$

Fig. 23. New scaling circuits — Improved Oldham fractal chain circuits after scaling extension: Normalized case

■ High-frequency validity and inverse proportion extension

By scaling the half-order high frequency valid improved Oldham fractal chain circuit (see fig. 9*(b)*), we get a new scaled fractal chain circuit as shown in fig.23 *(b)*, whose scaling equation is

$$y(w) = \frac{1}{2w} + \frac{1}{1 + \dfrac{1}{\dfrac{1}{2w} + \alpha y(\varrho w)}}. \tag{44}$$

Obviously, this is an irregular scaling equation which is difficult or impossible to solve analytically. Maybe we need to introduce a new special function? However, in the high frequency range, a regular scaling equation can be approximated with Liu's rough solution:

$$y(w) \approx \alpha y(\varrho w) \Longrightarrow y(w) \approx y_{\mathrm{Liu}}(w) = \kappa w^{\mu_{\mathrm{Liu}}}, \qquad \mu_{\mathrm{Liu}} = -\lg\alpha \,/\, \lg\varrho. \tag{45}$$

This conclusion shows that the corresponding circuit **should be** a high frequency valid scaled FAC!

Why do we say "should be" here? Because for the scaled circuit to be valid in the high frequency range, the following conditions must be met:

$$0 < \alpha < 1,\ 0 < \beta < 1,\ 0 < \varrho = \alpha\beta < 1. \tag{46}$$

Otherwise, each subsection of this scaled circuit will operate in the low frequency band showing negative first-order operational performance (i.e., almost pure capacitive operational performance), and in the high frequency band is showing resistive operational performance!

Scaling extension that satisfies condition (44) is called inverse proportion extension (IPE).

Now, we can preliminarily summarize the following conclusions:

LFV→direct proportion extension→LFV;

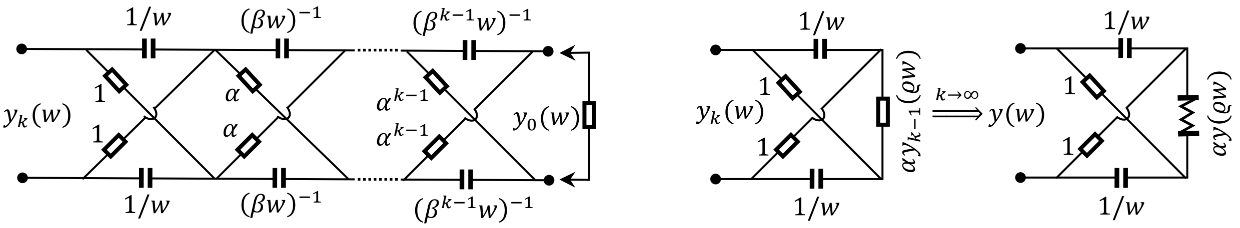
HFV→inverse proportion extension→HFV.

■ Full-frequency validity and scaling extension

So far, we have obtained five **Full Frequency Valid** (FFV), ideal approximation half-order FACs:

★ Carlson fractal lattice circuit (1960, Carlson G.E., see fig.10) [14],
★ Nakagawa-Sorimach fractal tree circuit (1992, Nakagawa M., Sorimach K., fig.11) [22],
★ Pu fractal pyramid circuit (2005, Pu Yifei, Yuan Xiao,et al, fig.12) [37],
★ Pu fractal tree circuit (2005, Pu Yifei, Yuan Xiao, et al, fig.12) [38, 39],
★ Yuan fractal tree circuit (2013, Yuan Xiao et al, fig.13) [1, 40].

Considering negative half-order, FFV Carlson fractal lattice circuit (see fig.10), its scaling extension case is shown in fig.24. The irregular scaling equation corresponding to this scaling circuit,



(a) Prototype circuit *(b)* Scaled iterating circuit and scaling equation circuit

Fig. 24. Scaled fractal lattice circuits: Normalized case

which we call the lattice scaling equation, [2, 40, 50] is

$$y(w) = F_{\mathrm{C}}\big(\alpha y(\varrho w)\big),\ F_{\mathrm{C}}(x)=\frac{2+(1+w)x}{(1+w)+2wx} \quad \begin{pmatrix} \alpha \neq 1, \beta \neq 1 \\ \varrho = \alpha\beta \neq 1 \end{pmatrix}. \tag{47}$$

According to the structure of the scaling fractal lattice circuit, lattice scaling equation (47) is approximately solved, and Liu's rough solution is obtained:

$$y(w) \approx y_{\mathrm{Liu}}(w) = \kappa w^{\mu_{\mathrm{Liu}}}, \qquad \mu_{\mathrm{Liu}} = -\frac{\lg\alpha}{\lg\sigma}.$$

From this we can conclude: direct proportion extension→LFV; inverse proportion extension→HFV.

Using Liu's rough solution can only preliminarily judge the operational validity. The actual operational characteristic curves of the scaling lattice circuit are shown in Fig.25!

6.3. Continued fraction expansion approximations for half-order operator and strange scaling equations

Consider the mathematical basis of the ancient Continued Fraction Expansion approximations of square root — classical identity relations:

$$\begin{cases} \sqrt{1+w^{\pm 1}} = 1 + \dfrac{w^{\pm 1}}{1+\sqrt{1+w^{\pm 1}}}, \\ \sqrt{1+w^{\pm 1}} - 1 = \dfrac{w^{\pm 1}}{2+\sqrt{1+w^{\pm 1}}-1}, \\ \sqrt{w^{\pm 1}} = \dfrac{w^{\pm 1}+\sqrt{w^{\pm 1}}}{1+\sqrt{w^{\pm 1}}}. \end{cases} \tag{48}$$

Of course, there could be other identities. Let $\sqrt{1+w^{\pm 1}} = y(w)$, $\sqrt{1+w^{\pm 1}} - 1 = y(w)$ (to non-ideal approximation), and $\sqrt{w^{\pm 1}} = y(w)$ (to an ideal approximation) respectively, we obtain three simple, half-order valid algebraic iterating equations:

$$y(w) = F_{\mathrm{CFE1}}\big(y(w)\big), \qquad y(w) = F_{\mathrm{CFE2}}\big(y(w)\big), \qquad y(w) = F_{\mathrm{CFE3}}\big(y(w)\big); \tag{49}$$

$$F_{\mathrm{CFE1}}(x) = 1 + \frac{w^{\pm 1}}{1+x}, \qquad F_{\mathrm{CFE2}}(x) = 1 + \frac{w^{\pm 1}}{1+x}, \qquad F_{\mathrm{CFE3}}(x) = \frac{w^{\pm 1}+x}{1+x}.$$

It is easy to verify that the three iterating processes thus constructed satisfy the computational rationality and positive reality principle. Their operational validity is self-evident!

In mathematics, by direct scaling extension, we get three sets of irregular strange scaling equations:

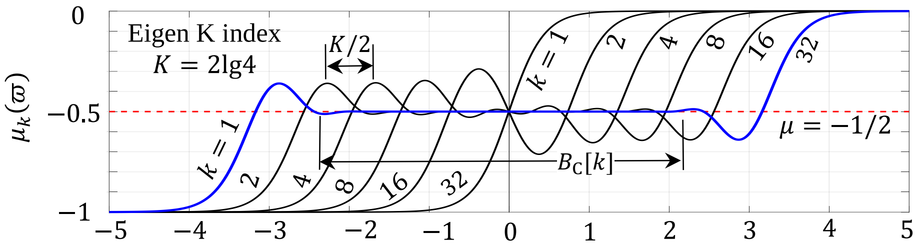
$$y(w) = F_{\text{CFE1}}\big(\alpha^{\pm 1} y(\varrho w)\big) = 1 + \frac{w^{\pm 1}}{1+\alpha^{\pm 1} y(\varrho w)}\ , \tag{50a}$$

$$y(w) = F_{\text{CFE2}}\big(\alpha^{\pm 1} y(\varrho w)\big) = \frac{w^{\pm 1}}{2+\alpha^{\pm 1} y(\varrho w)}\ , \tag{50b}$$

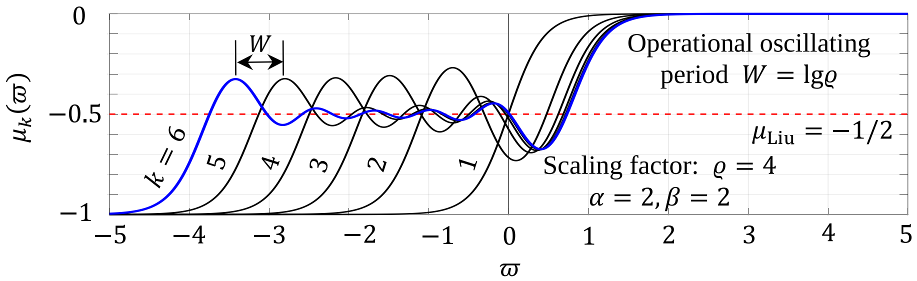
$$y(w) = F_{\text{CFE3}}\big(\alpha^{\pm 1} y(\varrho w)\big) = \frac{w^{\pm 1}+\alpha^{\pm 1} y(\varrho w)}{1+\alpha^{\pm 1} y(\varrho w)}\ . \tag{50c}$$

As we shall see, these three sets of equations contain very rich connotations and also have some unusual characteristics. For example, to select (50c), consider the following equation:

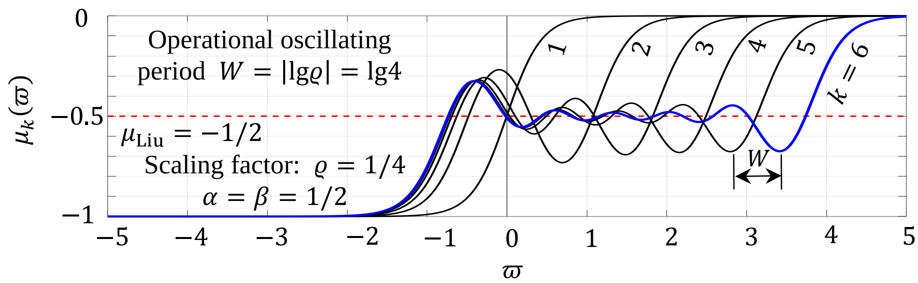
$$y(w) = \frac{w+\alpha^{-1} y(\varrho w)}{1+\alpha^{-1} y(\varrho w)}. \tag{6.12}$$



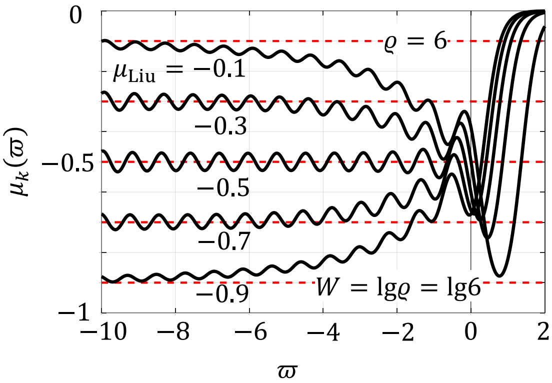
(a) Before scaling extension: $\alpha = \beta = 1$, $\mu = -1/2$



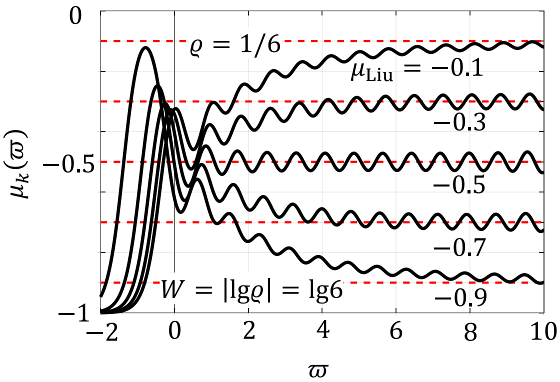
(b) Direct proportion extension: $\alpha = \beta = 2$, $\varrho = 4$, $\mu_{\text{Liu}} = -1/2$



(c) Inverse proportion extension: $\alpha = \beta = 1/2$, $\varrho = 1/4$, $\mu_{\text{Liu}} = -1/2$



(d) Direct proportion extension: LFH, $\varrho = 6$



(e) Inverse proportion extension: HFV, $\varrho = 1/6$

Fig. 25. Scaling fractal lattice circuits: Normalized case

In the extreme frequency range, we have

$$\frac{1}{\alpha}y(\varrho w) \approx y(w) \xleftarrow[\text{Low frequency}]{0\leftarrow|w|<1} y(w) = \frac{w + \alpha^{-1}y(\varrho w)}{1 + \alpha^{-1}y(\varrho w)} \xrightarrow[\text{High frequency}]{1<|w|\to\infty} y(w) \approx \frac{w}{\alpha^{-1}y(\varrho w)}. \tag{52}$$

Therefore, in the low frequency range, by direct proportion extension, a regular scaling equation is approximately obtained and has an analytical Liu's rough solution, that is

$$y(w) \approx \frac{1}{\alpha}y(\varrho w) \Longrightarrow y(w) \approx \kappa w^{\mu_{\text{Liu}}}, \qquad \mu_{\text{Liu}} = \frac{\lg\alpha}{\lg\varrho}, \ \begin{pmatrix} 1 < \alpha < \infty \\ 1 < \varrho < \infty \end{pmatrix}. \tag{53}$$

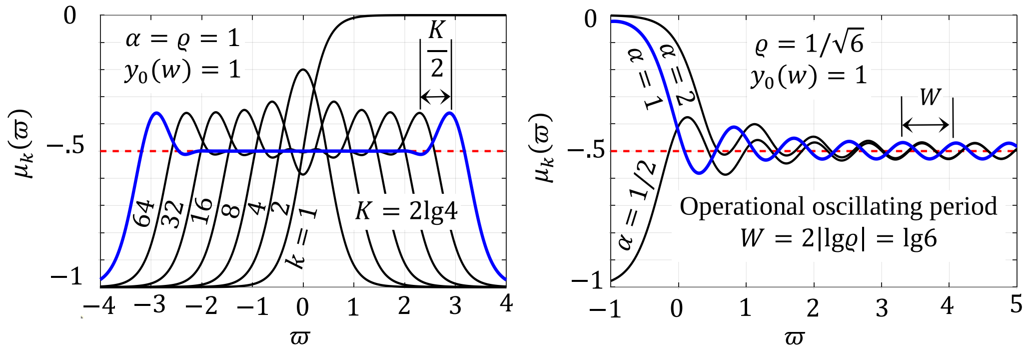
In the high frequency range, by inverse proportion extension, a quasi-regular scaling equation is approximated and has a peculiar analytical rough solution,

$$y(w) \approx \frac{\alpha w}{y(\varrho w)} \Longrightarrow y(w) \approx \sqrt{\alpha}/\sqrt[4]{\varrho}w^{\frac{1}{2}}, \ \begin{pmatrix} 0 < \alpha < \infty \\ 0 < \varrho < 1 \end{pmatrix}. \tag{54}$$

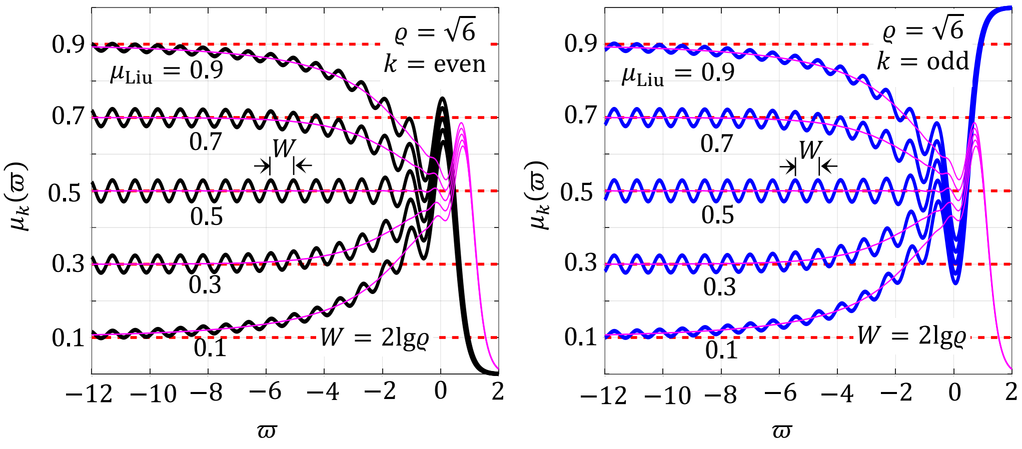
Here, after scaling extension, it is still half-order valid, and its operational order is independent of α and ϱ! This is one of the reasons why we call equation (51) etc., as strange scaling equation. From this, we can also obtain a strange scaling equation which only deterimined by the scaling factor ϱ, that is

$$y(w) = \frac{w + y(\varrho w)}{1 + y(\varrho w)}. \tag{55}$$

The order-frequency characteristic curves of a set of actual solutions of the strange scaling equation (51) are shown in fig. 26. In fig. 26*(c)*, when the number of iterations *k* is adjacent to an even number and an odd number, the phase of the operational oscillations in the approximation frequency band almost differ by π radians. Therefore, by taking advantage of this special property, the corresponding circuits of the two cases are connected in parallel, which will greatly cancel the operational oscillating effects. See the fine magenta curves in fig. 26*(c)*.



(a) Before scaling extension *(b)* Inverse proportion extension



(c) Direct proportion extension: $\varrho = \sqrt{6}$, $\mu_{\text{Liu}} = \lg\alpha\,/\lg\varrho$, $y_0(w) = 1$

Fig.26. The order-frequency characteristic curves of a set of actual solutions of equation (49): $y_0(w) = 1$

The strange scaling equations and their corresponding rational iterating processes have a lot of hidden content.

6.4. Limit-asymptotic behaviors of scaling FACs and their Irregular scaling equations

For scaling FACs, there is always, in general, an inherent quasi-periodic operational oscillating effect. This is because their corresponding irregular scaling equations, in mathematical form, can be written as follows:

$$y(w) = F\big(\alpha^{\pm 1} y(\varrho^{\pm 1} w)\big) \xrightarrow[y(\mathrm{j}\cdot 10^{\varpi}) = \eta(\varpi)]{w = \mathrm{j}\cdot 10^{\varpi}} \eta(\varpi) = F\big(\alpha^{\pm 1}\eta(\varpi \pm \lg\varrho)\big). \tag{56}$$

The period $W \propto \lg\varrho$ and the intensity (i.e., amplitude) of this operational oscillation are almost positively related to the scaling factor ϱ. In the approximation frequency band, the oscillation is almost in the form of a sine wave with period W, and its amplitude can be analytically expressed very precisely in theory.

The appearance of quasi-periodic operational oscillations in the approximation frequency band destroys the desired constant phase and constant order properties !

In order to reduce the operational oscillating effect (mainly to reduce the intensity), from the point of view of mathematical theory, it must meet

$$W \propto |\lg\varrho| \to 0 \Leftrightarrow \begin{Bmatrix} \varrho = \alpha\beta \to 1 \\ \alpha \to 1, \beta \to 1 \end{Bmatrix}. \tag{57}$$

Which will reduce the approximation benefit. Under the condition that Liu's operational order relation is guaranteed, that is, $\mu_{\text{Liu}} = \pm \lg\alpha / \lg\varrho$, the scaled circuit exhibits a limit-asymptotic behavior (LAB).

For example, a Liu-Kaplan fractal chain FAC has the limit-asymptotic behavior shown in fig.6.6. In high frequency range, all the different operational orders in the low frequency, band are all conform to the half-order case! All order-roads lead to half-order. We are eager to see who can verify this theoretical conclusion experimentally, or find evidence in nature.

The study of LAB of scaled FACs naturally leads to the physical realization and fabrication of fractal-order distributed parameter elements or components by analogical transformation.

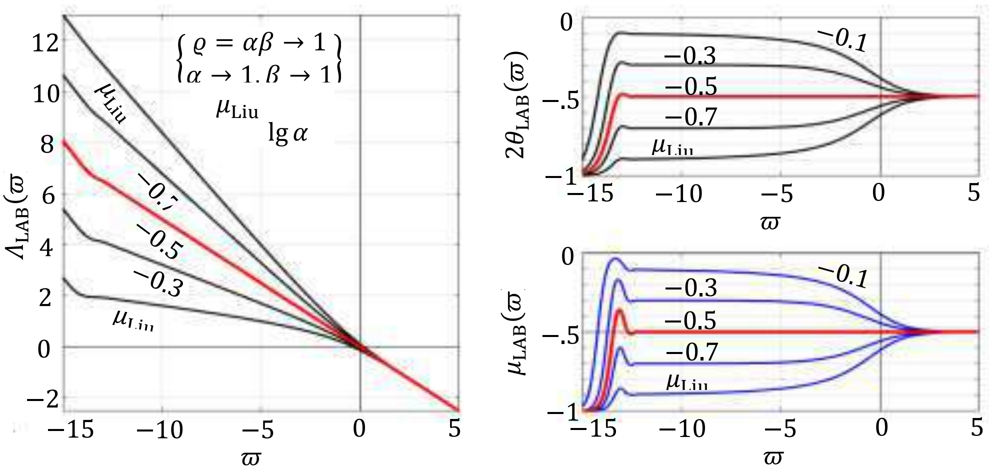


Fig. 27. Limit-asymptotic behavior (LAB) of Liu-Kaplan fractal chain circuit: normalized case

6.5 Scaling extension and irregular scaling equations

Some typical scaling extensions and their corresponding mathematical descriptions are listed in table 2.

Table 2. Some typical scaling extensions and their mathematical representations

<table>
<tr><th colspan="3" rowspan="2">Type and Name of FAC</th><th rowspan="2">Immittance function</th><th rowspan="2">Half-order iterating function $F(x)$</th><th rowspan="2">Scaling extension rule</th><th rowspan="2">Irregular Scaling Equation $y(w) = F(\alpha^{\pm 1} y(\varrho^{\pm 1} w))$</th><th colspan="2">Operational validity</th><th rowspan="2">Scaling equation: Type, Name</th></tr>
<tr><th>extension, valid band</th><th>$\mu_{\text{Liu}},\ W$</th></tr>
<tr><td rowspan="4">Morrison fractal FACs</td><td colspan="2">Parallel mode</td><td>Admittance</td><td>$\langle \frac{1}{1+\frac{1}{w}} + x \rangle$</td><td rowspan="2">$\alpha \in \mathbb{R}^+$ $\beta \in \mathbb{R}^+$ $\alpha \neq 1$ $\beta \neq 1$</td><td>$\frac{1}{1+\frac{1}{w}} + \frac{y(\varrho w)}{\alpha}$</td><td rowspan="2">DP→LF IP→HF</td><td>$\mu_{\text{Liu}} = \frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$</td><td rowspan="2">Hill scaling equation</td></tr>
<tr><td colspan="2">Series mode</td><td rowspan="3">Impedance</td><td>$\langle \frac{1}{1+w} + x \rangle$</td><td>$\frac{1}{1+w} + \alpha y(\varrho w)$</td><td rowspan="3">$\mu_{\text{Liu}} = -\frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$</td></tr>
<tr><td rowspan="4">Liu-Kaplan fractal chain</td><td>I</td><td>$1 + \frac{1}{w + \frac{1}{x}}$</td><td>LF→DP $1 \le \alpha$ $1 \le \beta$</td><td>$1 + \frac{1}{w + \frac{1}{\alpha y(\varrho w)}}$</td><td>DP→LF</td><td rowspan="4">Liu-Kaplan scaling equation</td></tr>
<tr><td>III</td><td>$\frac{1}{w} + \frac{1}{1 + \frac{1}{x}}$</td><td rowspan="2">HF→IP $0 < \alpha \le 1$ $0 < \beta \le 1$</td><td>$\frac{1}{w} + \frac{1}{1 + \frac{1}{\alpha y(\varrho w)}}$</td><td rowspan="2">IP→HF</td></tr>
<tr><td rowspan="2"></td><td>II</td><td rowspan="2">Admittance</td><td>$1 + \frac{1}{\frac{1}{w} + \frac{1}{x}}$</td><td>$1 + \frac{1}{\frac{1}{w} + \frac{\alpha}{y(\varrho w)}}$</td><td rowspan="2">$\mu_{\text{Liu}} = \frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$</td></tr>
<tr><td>IV</td><td>$w + \frac{1}{1 + \frac{1}{x}}$</td><td>LF→DP $1 \le \alpha$ $1 \le \beta$</td><td>$w + \frac{1}{1 + \frac{\alpha}{y(\varrho w)}}$</td><td>DP→LF</td></tr>
</table>

Carlson Fractal lattce		Impedance	$\frac{2+(1+w)x}{1+w+2wx}$	FF $\alpha \in \mathbb{R}^+$ $\beta \in \mathbb{R}^+$	$\frac{2+(1+w)\alpha y(\varrho w)}{1+w+2w\alpha y(\varrho w)}$	DP→LF IP→HF	$\mu_{\text{Liu}} = -\frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$	Lattice scaling equation
Pu fractal tree			$\frac{2+(1+w)x}{1+w+1wx}$		$\frac{2+(1+w)\alpha y(\varrho w)}{1+w+w\alpha y(\varrho w)}$			Double scaling equation
Yuan fractal tree			$\frac{1+(1+w)x}{1+w+2wx}$		$\frac{1+(1+w)\alpha y(\varrho w)}{1+w+2w\alpha y(\varrho w)}$			
Charef rational appro-ximation	I	Immittance	$\langle\frac{1+w}{1+\frac{w}{\alpha}}x\rangle$	$\alpha \in \mathbb{R}^+$ $\beta \in \mathbb{R}^+$ $\alpha \neq 1$ $\beta \neq 1$	$\frac{1+w}{1+\frac{w}{\alpha}}y\left(\frac{w}{\varrho}\right)$	HF	$\mu_{\text{Liu}} = -\frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$	Neoteric scaling equation
	D		$\langle\frac{1+w/\alpha}{1+w}x\rangle$		$\frac{1+\frac{w}{\alpha}}{1+w}y\left(\frac{w}{\varrho}\right)$		$\mu_{\text{Liu}} = \frac{\lg\alpha}{\lg\varrho}$ $W = \lvert\lg\varrho\rvert$	
Square root's Continued Fraction Expansion	I		$1+\frac{w^{\pm1}}{1+x}$	$\alpha \in \mathbb{R}^+$ $\beta \in \mathbb{R}^+$	$1+\frac{w^{\pm1}}{1+\alpha^{\pm1}y(\varrho w)}$	$\mu_{\text{Liu}} = \pm\frac{1}{2}$ $W = 2\lvert\lg\varrho\rvert$		Strange scaling equation
	II		$\frac{w^{\pm1}}{2+x}$		$\frac{w^{\pm1}}{2+\alpha^{\pm1}y(\varrho w)}$			
	III		$\frac{w^{\pm1}+x}{1+x}$	FF $\alpha \in \mathbb{R}^+$ $\beta \in \mathbb{R}^+$	$\frac{w^{\pm1}+\alpha^{\pm1}y(\varrho w)}{1+\alpha^{\pm1}y(\varrho w)}$	$\mu_{\text{Liu}} = \pm\frac{\lg\alpha}{\lg\varrho}$		
Instructions		$\langle F(x)\rangle$ Invalid iterating		Scaling feature parameter: α, β Scaling factor: $\varrho = \alpha\beta$		μ_{Liu}: Liu's order W: Operational oscillating period		
		LF: Low-Frequency Validity; HF: High-Frequency Validity; FF: Full-Frequency Validity.		DP: Directly Proportion Extension; IP: Inverse Proportion Extension.				

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МАТЕМАТИЧЕСКИЕ ПРИНЦИПЫ ПОСТРОЕНИЯ СХЕМ ФРАКТАНСНОЙ АППРОКСИМАЦИИ И ИХ ПРИЛОЖЕНИЯ

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Аннотация: Данная статья написана к 100-летию со дня рождения Рашида Шакировича Нигматуллина. В начале 1960-х годов он впервые реализовал фрактальный элемент, выполняющий операции дифференцирования и интегрирования половинного порядка в электрохимии. В последние годы, когда теория и применение дробного исчисления стали горячей темой во многих областях, схемное и математическое моделирование сложных явлений и процессов дробного порядка, а также их физическая реализация и практическое применение схем и систем дробного порядка являются особенно важными и востребованными. Проектирование и построение схем фрактансной аппроксимации (СФА) являются эффективным методом реализации дробных операторов и дробных элементов. В этой статье мы представили и обсудили исследования и разработки в области СФА по следующим направлениям: 1) пионеры в исследовании СФА; 2) основные понятия об элементах схем дробного порядка и СФА; 3) фрактальные цепочечные схемы Олдема и их математические описания, некоторые классические фрактальные СФА половинного порядка; 4) математические основы анализа в частотной области – характеристики в частотной области и рабочие характеристики; 5) фрактальные цепочки Лю-Каплана и их математические описания; 6) теория масштабного расширения и нерегулярные уравнения масштабирования.

Ключевые слова: фрактальный элемент, фрактальный оператор, фрактанс, фрактор, фрактал, схемы фрактансной аппроксимации, рабочие характеристики, моделирование аналоговых схем, математическая рациональная аппроксимация, масштабное расширение, нерегулярные уравнения масштабирования.

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